12.2 Review

In 12.2, we learn some basics of vectors. Most notably:

1. When a vector if represented in the form $\mathbf{v} = \langle v_1, v_2 \rangle$, then that means if the tail is drawn at the origin, then the head would be located at the point (v_1, v_2). That same vector can be drawn with the tail in locations other than the origin. As long as the vector has the same length and same direction, then it is the same vector (no matter where you draw the tail). Same goes for vectors in $\mathbb{R}^3$ which we write as $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

2. If you are given a specific head and tail locations $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ and you want a vector that points from A to B, then you just subtract to get the vector

$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.$$

3. If you add the components of two vectors, then you get the 'resultant' sum vector. The vector $\mathbf{a} + \mathbf{b}$ can be visually seen by drawing $\mathbf{a}$ from the origin followed by drawing $\mathbf{b}$ from the head of $\mathbf{a}$, then the sum is what you get when you draw a vector from the origin to the resulting head location of $\mathbf{b}$.

   The vector sum (just discussed) and the vector difference (mentioned above) are visually shown below:

   For the vector sum visualized above: $\mathbf{a} = \langle 3, 2 \rangle$ and $\mathbf{b} = \langle -1, 2 \rangle$, so $\mathbf{a} + \mathbf{b} = \langle 2, 4 \rangle$.
   For the vector difference visualized above: $\mathbf{a} = \langle 3, 2 \rangle$ and $\mathbf{b} = \langle 2, 4 \rangle$, so $\mathbf{b} - \mathbf{a} = \langle -1, 2 \rangle$

4. Multiplying a vector by a constant (scalar) ultimately just changes the length of the vector. If the scalar is positive, the vector points in the same direction. If the scalar is negative, the vector points in the opposite direction. Here is a visualization of scalar multiplication. In the example below, I show the vector $\mathbf{a} = \langle 2, 1 \rangle$ along with the vectors $2\mathbf{a} = \langle 4, 2 \rangle$, $-\mathbf{a} = \langle -2, -1 \rangle$, and $-3\mathbf{a} = \langle -6, -3 \rangle$.
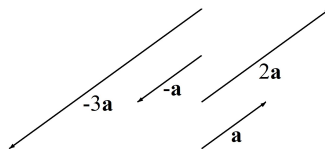
5. Another common (and equivalent) way we write vectors involves defining the standard basis vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$. We see by just substitution and adding that in this example $\langle 2, 3, -5 \rangle = 2\langle 1, 0, 0 \rangle + 3\langle 0, 1, 0 \rangle - 5\langle 0, 0, 1 \rangle = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$. So, in general, you often see vectors written in the following two equivalent ways:

$$\langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

6. We can get the length, or magnitude, of a vector by using the distance formula. And we use the following notation:

$$\text{In } \mathbb{R}^2 : \ |\mathbf{v}| = \sqrt{v_1^2 + v_2^2} \quad \text{and} \quad \text{In } \mathbb{R}^3 : \ |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Once you know the magnitude, we will often rescale our vector to length one (a vector of length one is called a unit vector). In other words:

$$\frac{1}{|\mathbf{v}|}\mathbf{v} = \ \text{a unit vector in the same direction as } \mathbf{v}.$$

7. If r is the magnitude of a vector $\mathbf{v} = \langle x, y \rangle$ and θ is the angle the vector makes with the positive x-axis, then you can use basic trigonometry to deduce that

$$x = r\cos(\theta) \ , \ y = r\sin(\theta) \ , \ x^2 + y^2 = r^2 \ , \ \tan(\theta) = \frac{y}{x}.$$

In many applications, you have access to information about r and θ and you use the basic facts above to find x and y in order to find the vector $\langle x, y \rangle$. Here is a picture:
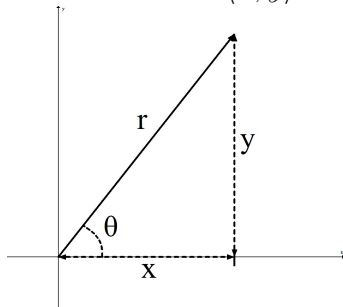
8. If you are given a slope in $\mathbf{R}^2$ and you want to give a vector with that same slope, you can quickly do it as follows:

$$\langle 1, m \rangle = \ \text{a vector that is parallel to any line with slope } m.$$

For example, the vector $\langle 1, 3 \rangle$ would be parallel to the line $y = 3x$ (as would any scale multiple of this vector, for example $\langle 2, 6 \rangle$ would be parallel to the line $y = 3x$).
Another example, the vector $\langle 1, \frac{1}{4} \rangle$ would be parallel to the line $y = \frac{1}{4}x$ (as would $\langle 4, 1 \rangle$).