

12.1 Review

In 12.1, we learn some basics of three-dimensional space. First a bit of terminology:

- \mathbb{R}^2 is the set of all points in two dimensional space, meaning the set of all pairs (x, y) where x and y are real numbers.
- \mathbb{R}^3 is the set of all points in three dimensional space, meaning the set of all triples (x, y, z) where x, y and z are real number.
- The **first octant** of \mathbb{R}^3 are all the points (x, y, z) where $x \geq 0, y \geq 0,$ and $z \geq 0.$

Most of the rest of this section is focused on working with distances in \mathbb{R}^3 and understanding \mathbb{R}^3 basics.

Some general notes:

1. If you are on the xy -plane, then you are on a point (x, y, z) where $z = 0.$ So the only condition (which is what we call the equation for this plane) is that $z = 0.$ We sometimes also write this condition in set notation, namely the xy -plane is the set of all points of the form $(x, y, 0)$ which can be written as $\{(x, y, 0) : x \text{ and } y \text{ are real numbers}\}.$
2. If you are on the xz -plane, then you are on a point (x, y, z) where $y = 0.$ So the equation is $y = 0.$
3. If you are on the yz -plane, then you are on a point (x, y, z) where $x = 0.$ So the equation is $x = 0.$
4. If you are on the x -axis, then you are on a point (x, y, z) where both $y = 0$ and $z = 0.$ So this axis is defined by the set of equation $y = 0$ and $z = 0.$
5. If you are on the y -axis, then you are on a point (x, y, z) where both $x = 0$ and $z = 0.$
6. If you are on the z -axis, then you are on a point (x, y, z) where both $x = 0$ and $y = 0.$

Distances:

1. The distance between two points (x_1, y_1) and (x_2, y_2) in \mathbb{R}^2 is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

(this is really just the Pythagorean theorem put to use).

2. The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in \mathbb{R}^3 is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

3. To find other distances, figure out the two points in question, then use the distance formula.

- The distance from (x_1, y_1, z_1) to the xy -plane is the same as asking the distance to $(x_1, y_1, 0).$ Using the distance formula (or common sense) will tell you this distance is just $z_1.$ Similarly, the distance from (x_1, y_1, z_1) to the xz -plane is $y_1.$ And the distance from (x_1, y_1, z_1) to the yz -plane is $x_1.$
- The distance from (x_1, y_1, z_1) to the x -axis is the same as asking the distance to $(x_1, 0, 0).$ The distance formula then gives $\sqrt{(x_1 - x_1)^2 + (y_1 - 0)^2 + (z_1 - 0)^2} = \sqrt{y_1^2 + z_1^2}.$ Similarly, the distance from (x_1, y_1, z_1) to the y -axis is $\sqrt{x_1^2 + z_1^2}.$ And the distance from (x_1, y_1, z_1) to the z -axis is $\sqrt{x_1^2 + y_1^2}.$

Spheres:

1. If you are given a 'center' point (x_0, y_0, z_0) and you want to describe all points that are at an equal distance r from that center point, then you are describing the a sphere (this word just means the points on the outer edge of the sphere). So you can just use the distance formula to describe all such points (x, y, z) that satisfy this description. Namely,

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r , \text{ usually written as } (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

This last equation is called standard form by our text.

2. So to find the equation of a sphere you need to find the radius r and the center (x_0, y_0, z_0) .
3. In order to get to standard form from an equation that is not in standard form, you need to know how to complete the square. Here is an example: The equation $2x^2 + 12x + 2y^2 + 2z^2 + 6z = 4$ is the equation for some sphere, but it is not in standard form. Here is the necessary algebra in order to get to standard form:

- (a) Divide all terms on both sides of the equation by 2 in order to get ones for coefficients of x^2 , y^2 and z^2 . This yields $x^2 + 6x + y^2 + z^2 + 3z = 2$.
- (b) A perfect square always has a third term that is half the middle term squared (think about that for a second and consider an example like $(t + 8)^2 = t^2 + 16t + 64$, where the 64 is exactly half the middle term squared). We want to make perfect squares, so we need to add the right amounts to both sides. Half of 6 squared would be $3^2 = 9$ and half of 3 squared would be $(\frac{3}{2})^2 = \frac{9}{4}$. So we add these values to both sides to get

$$x^2 + 6x + 9 + y^2 + z^2 + 3z + \frac{9}{4} = 2 + 9 + \frac{9}{4}$$

- (c) Now, if we have done everything correctly so far, then we can just factor and be done

$$(x + 3)^2 + y^2 + \left(z + \frac{3}{2}\right)^2 = \frac{53}{4}$$

- (d) And in this example, the center of the sphere is $(x_0, y_0, z_0) = (-3, 0, -\frac{3}{2})$ and the radius is $r = \sqrt{\frac{53}{4}}$.