

14.3 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **14.3 Partial Derivatives:** We gave the definition for partial derivatives of $z = f(x, y)$ as

$$\frac{\partial z}{\partial x} = f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \text{and}$$
$$\frac{\partial z}{\partial y} = f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

To compute $\frac{\partial z}{\partial x}$, we fix y and compute the derivative with respect to x (that is, we treat y as a constant and x as a variable).

Similarly, To compute $\frac{\partial z}{\partial y}$, we fix x and compute the derivative with respect to y (that is, we treat x as a constant and y as a variable).

Here are several examples:

$$z = 3x^2 + 4y^3 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = 6x \quad \frac{\partial z}{\partial y} = 12y^2$$

$$z = x^2y^3 + xy \quad \Rightarrow \quad \frac{\partial z}{\partial x} = 2xy^3 + y \quad \frac{\partial z}{\partial y} = 3x^2y^2 + x$$

$$z = \sin(3xy) + \frac{\ln(x)}{y} \quad \Rightarrow \quad \frac{\partial z}{\partial x} = 3y \cos(3xy) + \frac{1}{xy} \quad \frac{\partial z}{\partial y} = 3x \cos(3xy) - \frac{\ln(x)}{y^2}$$

$$z = 3xy^5e^{6x} \quad \Rightarrow \quad \frac{\partial z}{\partial x} = 3y^5e^{6x} + 18xy^5e^{6x} \quad \frac{\partial z}{\partial y} = 15xy^4e^{6x}$$

Graphically, we have the following interpretations:

$f_x(x, d) =$ ‘the slope of the tangent line when drawn on the plane $y = d$ ’ = ‘slope in the x direction’
 $f_y(c, y) =$ ‘the slope of the tangent line when drawn on the plane $x = c$ ’ = ‘slope in the y direction’

Here is another way to look at it:

- Imagine that $z = f(x, y)$ is a surface that you are hiking on. That is, you are hiking around mountains and valleys that look like the graph of $z = f(x, y)$.
- Now imagine that you are standing on the surface at a point with x and y coordinates $x = c$ and $y = d$.
 - Turn and face in the direction parallel to the direction of the positive x -axis.
 - If $f_x(c, d)$ is negative, then you will be walking down hill.
 - If $f_x(c, d)$ is positive, then you will be walking up hill.
 - In addition, $f_x(c, d)$ tells you how steep your hike will be in this direction.
 - Turn and face in the direction parallel to the direction of the positive y -axis.
 - If $f_y(c, d)$ is negative, then you will be walking down hill.
 - If $f_y(c, d)$ is positive, then you will be walking up hill.
 - In addition, $f_y(c, d)$ tells you how steep your hike will be in this direction.

You should also know how to compute higher-order partial derivatives because we will be using them later in the course (specifically in 14.7). Make sure to be careful about the notation

$$\begin{aligned}f_{xx} &= (f_x)_x = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2}, \\f_{yy} &= (f_y)_y = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2}, \\f_{xy} &= (f_x)_y = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x}, \\f_{yx} &= (f_y)_x = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y}.\end{aligned}$$