Taylor Notes 4 and 5 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **Essentials** - After TN 4 and 5, you should be able to answer questions pertaining to the following three concepts.
   (a) Know the basic infinite Taylor series on page 17 and in the examples of the Taylor Notes, when they converge and where they come from. Also, know the basics of infinite Taylor series.
   (b) Be able to find new Taylor series by substituting and manipulating the basic series that you know.
   (c) Be able to find Taylor series for integrals and derivatives. And understand what it means to integrate/differentiate a Taylor series.

2. **The Basis Series** - In general, we defined the Taylor series for $f(x)$ based at b as $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n\to\infty} T_n(x)$ provided that the limit exists. This last part ("provided that the limit exist") is very important. So you need to be careful which values of x that you can use.

   The good news is that we can get many series that we want by simply using a few basic ones that we already know. I will expect you to know the following series (the convergence conditions are given at the right of each series):

$$
\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{1}{k!} x^k &&= 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots &&\text{, valid for all } x \text{ values.}\\
\cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} &&= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots &&\text{, valid for all } x \text{ values.}\\
\sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} &&= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots &&\text{, valid for all } x \text{ values.}\\
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k &&= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots &&\text{, for } -1 < x < 1.\\
\tan^{-1}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} &&= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots &&\text{, for } -1 < x < 1.\\
-\ln(1-x) &= \sum_{k=1}^{\infty} \frac{1}{k} x^k &&= x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \cdots &&\text{, for } -1 < x < 1.
\end{aligned}
$$

3. **Substituting and Manipulating Taylor Series** - It is a useful skill to be able to find new Taylor series from the the six given above. Here are a few ways that we do this:
   - Many problems simply involve changing the input (i.e. changing x). Just remember to replace x everywhere, including the convergence condition. Here are some particular things to watch out for:
     - Use Trig identities when needed. You may need the following

       (1) $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ (3) $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

       (2) $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ (4) $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
     - You may need to use partial fractions (remember this from Math 125). Remember that you write it in the appropriate form and solve for your coefficients. For example, $\frac{x}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2}$ and then you find A and B.

- If you ever get stuck, write out the first few terms of the series and work out the appropriate manipulations. Then see if you can find the formula for the rest of the coefficients.

- Below are three extra examples for you to work through (these are all based at $b = 0$):

$$\begin{aligned} \frac{1}{1+\frac{1}{8}x^3} &= \sum_{k=0}^{\infty}(-\frac{1}{8}x^3)^k \quad \text{, for } -1 < -\tfrac{1}{8}x^3 < 1. \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{8^k}x^{3k} \quad \text{, for } -2 < x < 2. \end{aligned}$$

$$\begin{aligned} \cos^2(3x) &= \frac{1}{2}(1+\cos(6x)) = \frac{1}{2}+\frac{1}{2}\cos(6x) \quad \text{, for all } x. \\ &= \frac{1}{2}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(6x)^{2k} = \frac{1}{2}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}6^{2k}x^{2k} \quad \text{, for all } x. \end{aligned}$$

$$\begin{aligned} \frac{1}{1-2x}+\frac{2}{1+3x} &= \sum_{k=0}^{\infty}(2x)^k + 2\sum_{k=0}^{\infty}(-3x)^k \quad \text{, for } -1 < 2x < 1 \text{ and } -1 < -3x < 1. \\ &= \sum_{k=0}^{\infty}2^kx^k + 2\sum_{k=0}^{\infty}(-3)^kx^k \quad \text{, for } -\tfrac{1}{3} < x < \tfrac{1}{3}. \\ &= \sum_{k=0}^{\infty}\left(2^k+2(-3)^k\right)x^k \quad \text{, for } -\tfrac{1}{3} < x < \tfrac{1}{3}. \end{aligned}$$

  (The convergence condition on the last example is valid because it is the only interval where they both series converge, it is the smaller interval).

4. **Changing Bases -** In most cases, we base our Taylor series at $b = 0$. However, occasionally it is useful to use other bases. A few problems in HW 4 give you practice with this. I think you will find that the following steps (*in this order*) works best to deal with these problems:

   (a) First find the Taylor series based at zero for the function in question.

   (b) Then replace x with $x - b$.

   (c) Simplify the function and solve to get what you want.

   Here is an example: Find the Taylor series of $f(x) = e^{2x}$ based at $b = 1$. Here are the steps:

   (a) First find the Taylor series based at zero:

   $$e^{2x} = \sum_{k=0}^{\infty}\frac{1}{k!}(2x)^k = \sum_{k=0}^{\infty}\frac{1}{k!}2^kx^k 1 + 2x + \frac{1}{2!}2^2x^2 + \frac{1}{3!}2^3x^3 + \cdots,$$

   which is valid for all x values.

   (b) Now replace x with $x - 1$:

   $$e^{2(x-1)} = \sum_{k=0}^{\infty}\frac{1}{k!}2^k(x-1)^k$$

(c) Now solve for e^{2x} to get the answer.

$$\begin{aligned}
e^{2(x-1)} &= \sum_{k=0}^{\infty} \frac{1}{k!} 2^k (x-1)^k \\
e^{2x-2} &= \sum_{k=0}^{\infty} \frac{1}{k!} 2^k (x-1)^k \\
e^{2x} e^{-2} &= \sum_{k=0}^{\infty} \frac{1}{k!} 2^k (x-1)^k \\
e^{2x} \tfrac{1}{e^2} &= \sum_{k=0}^{\infty} \frac{1}{k!} 2^k (x-1)^k \\
e^{2x} &= e^2 \sum_{k=0}^{\infty} \frac{1}{k!} 2^k (x-1)^k = e^2 \left(1 + 2(x-1) + \frac{1}{2!} 2^2 (x-1)^2 + \frac{1}{3!} 2^3 (x-1)^3 + \cdots \right).
\end{aligned}$$

5. **Integrating and Differentiating Series -** The truly great news about Taylor series is that they are 'easy' to integrate and differentiate. For example, I imagine that few students would have difficulty finding the derivative of $1 + x + x^2 + x^3$, which is $1 + 2x + 3x^2$, or finding the antiderivative of $1+7x^3+x^5$, which would be $x+\frac{7}{4}x^4+\frac{1}{6}x^6+C$. It is no more difficult to compute derivatives and antiderivatives for Taylor series. Here are a few other things to remember:

   - The open interval of convergence is unchanged by integration and differentiation. That is, the convergence condition comes along for the ride when you integrate or differentiate.

   Below are two extra examples for you to work through (these are all based at $b = 0$). Note that these integrals would be difficult/impossible for you to evaluate using Math 125 methods:

$$\begin{aligned}
\int_0^x \frac{1}{1-t^7}\, dt &= \int_0^x \sum_{k=0}^{\infty} (t^7)^k \, dt \text{ , for } -1 < x^7 < 1. \\
&= \int_0^x \left(1 + t^7 + t^{14} + t^{21} + \cdots \right) dt \text{ , for } -1 < x < 1. \\
&= \left(x + \frac{1}{8}x^8 + \frac{1}{15}x^{15} + \frac{1}{22}x^{22} + \cdots \right) \text{ , for } -1 < x < 1. \\
&= \sum_{k=0}^{\infty} \frac{1}{7k+1} x^{7k+1} \text{ , for } -1 < x < 1.
\end{aligned}$$

$$\begin{aligned}
\int_0^x t e^{t^{10}}\, dt &= \int_0^x t \sum_{k=0}^{\infty} \frac{1}{k!} t^{10k} \, dt \text{ , for all } x. \\
&= \int_0^x t \left(1 + t^{10} + \frac{1}{2!}t^{20} + \frac{1}{3!}t^{30} + \cdots \right) dt \text{ , for all } x. \\
&= \int_0^x \left(t + t^{11} + \frac{1}{2!}t^{21} + \frac{1}{3!}t^{31} + \cdots \right) dt \text{ , for all } x. \\
&= \frac{1}{2}x^2 + \frac{1}{12}x^{12} + \frac{1}{22}\frac{1}{2!}x^{22} + \frac{1}{32}\frac{1}{3!}x^{32} + \cdots \text{ , for all } x. \\
&= \sum_{k=0}^{\infty} \frac{1}{k!(10k+2)} x^{10k+2} \text{ , for all } x.
\end{aligned}$$

6. **Other Comments** You need to have an experimental attitude. Try writing out several terms and manipulate different parts until you get what you want. Also, make sure to read through all the examples in the Taylor notes. The more examples you can see the better.

   I hope that you can glimpse some of the surprising and beautiful results that we get from Taylor series. For instance, since $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ for all x, we have

$$e^1 = \sum_{k=0}^{\infty} \frac{1}{k!} 1^k = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

In other words, if you add up more and more of these terms, then you get closer to the actual value of e.

As another example,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots = \frac{1}{1 - 1/3} = 3/2$$

here we just used the result $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ which is valid for $\frac{1}{3}$ because it is between -1 and 1. So if you add up 1, then 1/3, then 1/9, then 1/27, and so on, you get closer and closer to 3/2=1.5.

I think this is remarkable, and it is your first glimpse of the beauty of advanced mathematics. Congratulations, you've stepped further into the land of the infinite.