1. (12 pts) A particle is moving in such a way that it's acceleration is given by $\mathbf{a}(t) = \langle 4, \sin(t), e^t \rangle$. The initial velocity is $\mathbf{v}(0) = \langle -6, 2, 0 \rangle$ and the initial position is $\mathbf{r}(0) = \langle 0, 0, 10 \rangle$.

(a) (5 pts) Find the curvature, κ, at time $t = 0$.

$$\vec{r}\,'(0) = \langle -6, 2, 0 \rangle$$
$$\vec{r}\,''(0) = \langle 4, 0, 1 \rangle$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & 2 & 0 \\ 4 & 0 & 1 \end{vmatrix} = \langle 2-0, -(-6-0), 0-8 \rangle$$

$$\kappa(0) = \frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|^3} = \frac{\sqrt{4+36+64}}{(36+4+0)^{3/2}} = \boxed{\frac{\sqrt{104}}{40^{3/2}}}$$

$$\approx 0.0403112$$

(b) (7 pts) Find the (x, y, z) coordinates of the particle at time $t = 2$ seconds. (You can leave your answers in exact form.) vfill

$$\vec{v}(t) = \langle 4t + c_1, -\cos(t) + c_2, e^t + c_3 \rangle$$

$$\vec{v}(0) = \langle -6, 2, 0 \rangle \Rightarrow c_1 = -6, \quad -1 + c_2 = 2, \quad 1 + c_3 = 0$$
$$c_2 = 3 \qquad c_3 = -1$$

$$\vec{v}(t) = \langle 4t - 6, -\cos(t) + 3, e^t - 1 \rangle$$

$$\vec{r}(t) = \langle 2t^2 - 6t + d_1, -\sin(t) + 3t + d_2, e^t - t + d_3 \rangle$$

$$\vec{r}(0) = \langle 0, 0, 10 \rangle \Rightarrow d_1 = 0, \ d_2 = 0, \ 1 + d_3 = 10$$
$$d_3 = 9$$

$$\vec{r}(2) = \langle 2(2)^2 - 6(2), -\sin(2) + 3(2), e^2 - 2 + 9 \rangle$$
$$= \boxed{\langle -4, 6 - \sin(2), e^2 + 7 \rangle}$$

2. (The two problems below are unrelated)

(a) (8 pts) Find the linearization $L(x,y)$ of $f(x,y) = \underline{\underline{ln}}(y) + e^{3x}\sqrt{xy+4y^2}$ at $(0,1)$.

$$f_x(x,y) = 3e^{3x}\sqrt{xy+4y^2} + e^{3x}\frac{y}{2\sqrt{xy+4y^2}}$$

$$f_x(0,1) = 3\sqrt{0+4} + \frac{1}{2\sqrt{0+4}} = 6 + \frac{1}{4} = \frac{25}{4}$$

$$f_y(x,y) = \frac{1}{y} + e^{3x}\frac{(x+8y)}{2\sqrt{xy+4y^2}}$$

$$f_y(0,1) = \frac{1}{1} + \frac{(0+8)}{2\sqrt{0+4}} = 1 + \frac{8}{4} = 3$$

$$f(0,1) = \ln(1) + \sqrt{0+4} = 2$$

$$\boxed{L(x,y) = 2 + \frac{25}{4}(x-0) + 3(y-1)}$$

(b) (8 pts) Let $f(x,y) = \dfrac{9}{x} + 3xy - y^2$. Find and classify all critical points of $f(x,y)$. (Classify using appropriate partial derivative tests).

$$f_x(x,y) = -\frac{9}{x^2} + 3y \overset{?}{=} 0 \implies y = \frac{3}{x^2}$$

$$f_y(x,y) = 3x - 2y \overset{?}{=} 0 \implies y = \frac{3}{2}x$$

Combine $\implies \frac{3}{x^2} = \frac{3}{2}x \implies 3 = \frac{3}{2}x^3 \implies 2 = x^3$

So $x = 2^{1/3}$ and $y = \frac{3}{2}2^{1/3}$

$f_{xx} = \frac{18}{x^3}$, $f_{yy} = -2$, $f_{xy} = 3$

$f_{xx}(2^{1/3}, \frac{3}{2}2^{1/3}) = \frac{18}{2} = 9$

$D = 9\cdot(-2) - 3^2 = -27 < 0$

$$\boxed{(x,y) = \left(2^{1/3}, \frac{3}{2}2^{1/3}\right), \quad \text{SADDLE}}$$

3. (a) (7 pts) Set up and evaluate a double integral to find the volume of the solid below the surface $z - 3x^2y = 0$ and above the triangle with vertices $(0,0)$, $(1,2)$, and $(0,2)$.

$$\int_0^1 \int_{2x}^{2} 3x^2 y \, dy \, dx$$

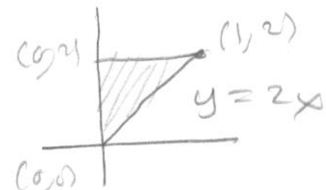
$$\int_0^1 \left(\frac{3}{2} x^2 y^2 \Big|_{2x}^{2} \right) dx$$

$$\int_0^1 \left(\frac{3}{2} x^2 (2)^2 - \frac{3}{2} x^2 (2x)^2 \right) dx$$

$$\int_0^1 6x^2 - 6x^4 \, dx$$

$$2x^3 - \frac{6}{5} x^5 \Big|_0^1$$

$$2 - \frac{6}{5} = \boxed{\frac{4}{5}}$$



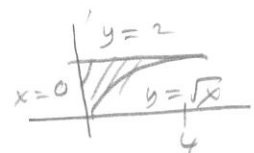
Also could do

$$\int_0^2 \int_0^{\frac{1}{2}y} 3x^2 y \, dx \, dy$$

(b) (7 pts) Evaluate the integral by reversing the order of integration: $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1} \, dy dx$.

$0 \le x \le 4$

$\sqrt{x} \le y \le 2$



$$\int_0^2 \int_0^{y^2} \frac{x}{y^5+1} \, dx \, dy$$

$$\int_0^2 \frac{1}{y^5+1} \left(\frac{1}{2} x^2 \Big|_0^{y^2} \right) dy$$

$$\frac{1}{2} \int_0^2 \frac{y^4}{y^5+1} \, dy$$

$u = y^5 + 1$

$du = 5y^4 dy$

$$\frac{1}{10} \int_1^{33} \frac{1}{u} \, du$$

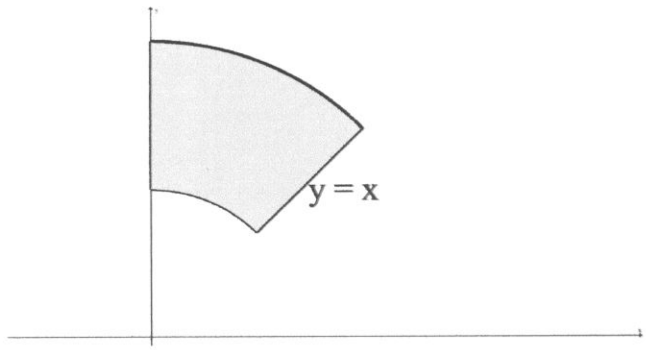
$$\frac{1}{10} \ln|u| \Big|_1^{33} = \boxed{\frac{1}{10} \ln(33)}$$

4. (8 pts) A lamina occupies the region R in the first quadrant that is above the line $y = x$ and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ (as shown below). The density is proportional to the distance from the origin. $\rho(x,y) = \sqrt{x^2+y^2}$

Find the y-coordinate of the center of mass, $\bar{y}$. (Give your final answer as a decimal to 4 digits).

$\pi/4 \le \theta \le \pi/2$

$1 \le r \le 2$



$\rho(x,y) = k\sqrt{x^2+y^2}$

TOTAL MASS $= \int_{\pi/4}^{\pi/2} \int_1^2 k r \, r \, dr \, d\theta = k \int_{\pi/4}^{\pi/2} d\theta \int_1^2 r^2 \, dr$

$= k\left(\frac{\pi}{2} - \pi/4\right)\left(\frac{1}{3} r^3 \Big|_1^2\right)$

$= \frac{k\pi}{4}\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{k\,7\pi}{12}$

$m_x = \iint_D y\rho(x,y)\, dA = k\int_{\pi/4}^{\pi/2} \int_1^2 r\sin\theta \, r \, r \, dr \, d\theta$

$= k\left(\int_{\pi/4}^{\pi/2} \sin\theta \, d\theta\right)\left(\int_1^2 r^3 \, dr\right)$

$= k\left(-\cos\theta \Big|_{\pi/4}^{\pi/2}\right)\left(\frac{1}{4} r^4 \Big|_1^2\right)$

$= k\left(-0 - -\frac{\sqrt{2}}{2}\right)\left(4 - \frac{1}{4}\right)$

$= \frac{k\sqrt{2}}{2} \cdot \frac{15}{4} = \frac{k\,15\sqrt{2}}{8}$

$\bar{y} = \frac{k\,15\sqrt{2}/8}{k\,7\pi/12} = \boxed{\frac{45\sqrt{2}}{14\pi} \approx 1.446936}$