Worksheet: Taylor Series Applications

As you have seen, Taylor series can be used to give polynomials that approximate a function around some point. Typically at first blush students often say Taylor series are odd and tedious to work with. And they ask "why would anyone ever do this?!".

The truth is, after a bit of practice, Taylor series are very easy to work with and are often much preferable to the *messy* function they are approximating. Taylor series have wide reaching applications across mathematics, physics, engineering and other sciences. And the concept of approximating a function, or data, using a series of function is a fundamental tool in modern science and in use in data analysis, cell phones, differential equations, *etc.*. Taylor series give you a first glimpse into this world of approximation (some other common approximation methods are Fourier series and wavelets, if you are curious you could look those up and read about them).

Here are a few quick applications of Taylor series that I hope help you better appreciate the tools you've learned.

1. (The Normal Curve) In probability we often talk about the normal bell curve. It is the most prominent probability distribution in statistics and it comes up in the study of data that is clustered around a single mean value. The standard normal distribution with mean 0 and standard deviation 1 is given by the probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

So assume $x = 0$ is the mean of your data (the mean test score for instance) and $x = \pm 1$ is one standard deviation from the mean (you would have to scale your test scores to match this in practice, but hopefully you get the idea).

(a) Draw a rough sketch of this function.

We measure probabilities by taking areas under this graph. Notably

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx = 1.$$

You can figure verify this by integrating $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx$ with a clever trick from polar coordinates and finding the value to be $\sqrt{2\pi}$. (Simon did this with you). Using the normal distribution, the probability that a student scores within one standard deviation from the median is given by:

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$$

In Math 125, I did an example where we approximated this value with Simpson's rule. Today we will approximate it in a better way in get the 'exact' answer (exact in the sense that it will be given as an infinite series).

(b) Give the Taylor series for $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ based at $b = 0$.

(c) Integrate your Taylor series to get the value of $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$. Put your answer in Sigma notation.

(d) Go far enough out in the sum to be confident you have the answer to 8 decimal digits.

2. (Simplifying *Messy* Functions) According the Einstein's theory of special relativity if an object at rest has mass m_0, then it's kinetic energy is given by

$$K = m_0c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right).$$

According to the theory of classic Newtonian physics, the kinetic energy is given by

$$K = \frac{1}{2}m_0v^2.$$

(a) By differentiating with respect to v, find the first two nonzero terms of the Taylor series for $\frac{1}{\sqrt{1-v^2/c^2}}$ based at zero. Then show that if you only use these terms, that the two formulas are the same (for velocities much smaller than the speed of light, the higher order terms are negligible, so these formulas are about the same for small velocities). There are many other examples like this. Read problems from the end of Section 11.11 of your book for more examples.

3. (Solving Differential Equations) If I told you that $y = ax^2 + bx + c$ was a solution to $y' = 6x + 7$ you could differentiate y and compare coefficients to get $2ax + b = 6x + 7$. From that you could conclude that $2a = 6$, so $a = 3$ and $b = 7$. Thus, $y = 3x^2 + 7x + c$ (but you already knew this). This same idea can be used to find solution to more complicated differential equations using Taylor series.

Here is an example: We know, from Math 125, how to solve the initial value problem $\frac{dy}{dx} = y$ with $y(0) = 1$. You might remember the solution. Now let's solve it another way.

Let $y = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_nx^n$ be the unknown Taylor series for $y(x)$. Since $y(0) = 1$, we know that $a_0 = 1$.

Differentiating we get, $y' = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}$. Putting these in the differential equation $\frac{dy}{dx} = y$, we get $a_1 + 2a_2x + 3a_3x^2 + \cdots = a_0 + a_1x + a_2x^2 + \cdots$. Now we equate coefficients to get

$$\begin{aligned} a_1 &= a_0 = 1, && \text{(by the initial condition).} \\ 2a_2 &= a_1 = 1, && \text{so } a_2 = \tfrac{1}{2}. \\ 3a_3 &= a_2 = \tfrac{1}{2}, && \text{so } a_3 = \tfrac{1}{6} = \tfrac{1}{3!}. \\ 4a_4 &= a_3 = \tfrac{1}{3!}, && \text{so } a_4 = \tfrac{1}{4!}. \end{aligned}$$

And, in general, we are seeing that $a_n = \frac{1}{n!}$. Thus, the answer is $y = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$, which we know from class is e^x. This may seem like a roundabout method, but it is very useful because it is a general method that can be used even when the differential equation is not separable.

(a) Try the method, on the following differential equation $y' = x + y$ with $y(0) = 0$. This is not separable, so you currently have no method from Math 125 to solve this. Let's try to solve it with Taylor series. Give the first 5 nonzero terms of the Taylor series for the answer using the method described in the previous example. You should recognize your answer. Rewrite your final answer in terms of known functions.

4. (Evaluating infinite sums) Most non-math folk wouldn't call this an application. But Dr. Loveless likes infinite series, so you must suffer his interest for a moment.

   (a) Give the values of the following sums by substituting into the appropriate Taylor series (assume they all converge):

   (i) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n$ (ii) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}$ (iii) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$

   (iv) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}$ (v) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ (vi) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$

   Aside: The last sum, (vi), can be used to approximate the value of π. It converges very slowly. That is, you have to go way out to get a lot of digits of accuracy, but there are adjustments you can make to get faster convergence. This has been a major method for approximating many digits of π in the past. (Read problem 7 from the Problem Plus of Chapter 11 for more details).

   (b) If there is time, try problem 5 from the Problem Plus of Chapter 15 (a double integral related to the sum of the inverse squares).

   Aside again: The sum $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, but the sums $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}, \sum_{n=1}^{\infty}\frac{1}{n^3}, \sum_{n=1}^{\infty}\frac{1}{n^4}$, *etc* are all known to converge. Finding explicit values for these sums has been a source of great interest in the history of mathematics. In 1644, Pietro Mengoli posed the problem of finding the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. In 1735, Leonhard Euler solved the problem and found that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. Euler is one of the greatest and most prolific mathematicians of all time. He published over 900 works spanning nearly all areas of math and science. He created many of the notations and conventions used today and for a good portion of his life he was blind. Our book discusses a solution to this particular problem in the problem plus (problems 5 and 6). We can't quite do problem 6 with the material we've covered, but you can read it if you are interested. Euler used a different method, but he still made use of Taylor series as part of his solution. Generalizing his solution, he found the explicit values for ALL even exponents. Explicit values for the odd exponents are still unknown and it is quite difficult to prove things about them.