Math 126 Basic Summary of Facts

Facts about Vectors, Curves and General 3D

Vector Basics:

$\lvert\mathbf{v}\rvert = \sqrt{v_1^2+v_2^2+v_3^2}$	$c\mathbf{v} = \langle cv_1, cv_2, cv_3\rangle$	$\frac{1}{\lvert\mathbf{v}\rvert}\mathbf{v}$ = 'unit vector in direction of $\mathbf{v}$
$\mathbf{u}+\mathbf{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3\rangle$	$\mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+u_3v_3$	$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
$\mathbf{u}\cdot\mathbf{v} = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos(\theta)$	$\mathbf{u}\cdot\mathbf{v} = 0$ means orthogonal	θ is the angle if drawn tail to tail
$\lvert\mathbf{u}\times\mathbf{v}\rvert = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\sin(\theta)$	$\mathbf{u}\times\mathbf{v}$ is orthogonal to both	$\lvert\mathbf{u}\times\mathbf{v}\rvert$ = parallelogram area
$\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert}$	$\mathbf{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert^2}\mathbf{a}$	

Comments: Know how to check/find vectors that are parallel or orthogonal. Be comfortable with computation, interpretations, and consequences.

Basic Lines, Planes and Surfaces (assume the constants a, b and c are positive in the last three rows):

Lines: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$	(x_0, y_0, z_0) = a point on the line $\langle a, b, c\rangle$ = a direction vector
Planes: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$	(x_0, y_0, z_0) = a point on the plane $\langle a, b, c\rangle$ = a normal vector
Cylinder: One variable 'missing'	Know basics of traces
Elliptical Paraboloid: $z = ax^2 + by^2$	Hyperbolic Paraboloid: $z = ax^2 - by^2$
Ellipsoid: $ax^2 + by^2 + cz^2 = 1$	Cone: $z^2 = ax^2 + by^2$
Hyperboloid of One Sheet: $ax^2 + by^2 - cz^2 = 1$	Hyperboloid of Two Sheets: $ax^2 + by^2 - cz^2 = -1$

Comments: You should be very good at finding lines/planes and naming basic shapes.

Basic Parametric and Polar in $\mathbb{R}^2$:

$\langle \frac{dx}{dt}, \frac{dy}{dt}\rangle$ = a tangent vector	$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
$x = r\cos(\theta)$	$y = r\sin(\theta)$
$x^2 + y^2 = r^2$	$\tan(\theta) = \frac{y}{x}$

Basic Parametric in $\mathbb{R}^3$:

$\mathbf{r}'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle$	$\mathbf{r}''(t) = \langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\rangle$
$\displaystyle\int \mathbf{r}(t)\,dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$	Note: There are three constants of integration.
Arc Length $= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$	$s(t) = \int_0^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\,du$
$\kappa(t) = \frac{\lvert\mathbf{r}'(t)\times\mathbf{r}''(t)\rvert}{\lvert\mathbf{r}'(t)\rvert^3}$	$\kappa(x) = \frac{\lvert f''(x)\rvert}{(1+f'(x)^2)^{3/2}}$ = '2D curvature'
$\mathbf{r}'(t) = \mathbf{v}(t)$ = velocity vector	$\lvert\mathbf{r}'(t)\rvert = \lvert\mathbf{v}(t)\rvert$ = speed
$\mathbf{r}''(t) = \mathbf{a}(t)$ = acceleration	$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt$ and $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt$
$\mathbf{T}(t) = \frac{1}{\lvert\mathbf{r}'(t)\rvert}\mathbf{r}'(t)$ = unit tangent	$\mathbf{N}(t) = \frac{1}{\lvert\mathbf{T}'(t)\rvert}\mathbf{T}'(t)$ = principal unit normal
$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\lvert\mathbf{r}'(t)\rvert} = \frac{d}{dt}\lvert\mathbf{r}'(t)\rvert$	$a_N = \frac{\lvert\mathbf{r}'(t)\times\mathbf{r}''(t)\rvert}{\lvert\mathbf{r}'(t)\rvert} = \kappa(t)(\lvert\mathbf{r}'(t)\rvert)^2$

Facts about Surfaces, Critical Points, and Double Integrals

Slopes on Surfaces.

Be able to find and graph the domain	Know the basics on level curves/contour maps
$f_x(x,y) = \frac{\partial z}{\partial x}$ = slope in x-direction	$f_y(x,y) = \frac{\partial z}{\partial y}$ = slope in y-direction
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$	Tangent plane/linearization.
$f_{xx}(x,y) = \frac{\partial^2 z}{\partial x^2}$ = concavity in x-direction	$f_{yy}(x,y) = \frac{\partial^2 z}{\partial y^2}$ = concavity in y-direction
$f_{xy}(x,y) = \frac{\partial^2 z}{\partial y \partial x}$ = mixed second partial	$f_{xy}(x,y) = f_{yx}(x,y)$ (Clairaut's Theorem)
$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ = measure of concavity	$D < 0$ means concavity changes (saddle)
$D > 0$, $f_{xx} > 0$ means concave up all directions	$D > 0$, $f_{xx} < 0$ means concave down all directions

Comments:

- To find critical points: Find f_x and f_y, set them BOTH equal to zero, then COMBINE the equations and solve for x and y.
- To classify critical points: Find f_{xx}, f_{yy}, and f_{xy}. At each critical point compute f_{xx}, f_{yy}, f_{xy} and D and make appropriate conclusions from the second derivative test.
- To find absolute max/min on a region: Find critical points inside the region. Then, over each boundary, substitution the xy-equation for the boundary into the surface to get a one variable function. Find the absolue max/min of the one variable function over each boundary. In the end, evaluate $f(x,y)$ at all the critical points inside the region and the critical numbers and endpoints (corners) on each boundary to find the largest and smallest output.

Volumes: $\displaystyle\iint_D f(x,y)\,dA$ = signed volume 'above' the xy-plane, 'below' $f(x,y)$ and inside the region D.

We also saw $\displaystyle\iint_D 1\,dA$ = area of D.

To set up a double integral:
(1) Solve for integrand ($z = f(x,y)$).
(2) Draw given xy-equations in the xy-plane. (label intersections).
(3) Draw xy-equations that occur from surface intersections.
(4) Set up the double integral(s) using the region for D.

Options for set up:

$\displaystyle\iint_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dydx$,	$y = g(x)$ = bottom,	$y = h(x)$ = top
$\displaystyle\iint_D f(x,y)\,dA = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\,dxdy$,	$x = p(y)$ = left,	$x = q(y)$ = right
$\displaystyle\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{w(\theta)}^{v(\theta)} f(r\cos(\theta), r\sin(\theta))\,rdrd\theta$,	$r = w(\theta)$ = inner,	$r = v(\theta)$ = outer

Center of Mass Application: If $\rho(x,y)$ = formula for density at a point in the region D, then

$$M = \text{total mass} = \iint_D \rho(x,y)\ dA\ ,\quad \overline{x} = \frac{1}{M}\iint_D x\rho(x,y)\ dA \quad \text{and} \quad \overline{y} = \frac{1}{M}\iint_D y\rho(x,y)\ dA$$

Facts about Taylor Polynomials and Error Bounds

$$T_1(x) = \sum_{k=0}^{1} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b).$$

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2.$$

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3.$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n.$$

Taylor inequalities

$$\text{ERROR} = |f(x) - T_1(x)| \le \frac{M}{2!}|x-b|^2 \qquad \text{, where } |f''(x)| \le M \text{ on the interval, and in general,}$$

$$\text{ERROR} = |f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x-b|^{n+1} \qquad \text{, where } |f^{(n+1)}(x)| \le M \text{ on the interval.}$$

Three types of error questions:

Given an interval $[b-a, b+a]$, find the $T_n(x)$ error bound:

1. Find $|f^{(n+1)}(x)|$.
2. Determine a bound (a *tight* bound if possible) for $|f^{(n+1)}(x)| \le M$ on the interval.
3. In Taylor's inequality $\frac{M}{(n+1)!}|x-b|^{n+1}$ replace M and replace x by an endpoint.

Find an interval so that $T_n(x)$ has a desired error:

1. Write $[b-a, b+a]$ and you will solve for a.
2. Find $|f^{(n+1)}(x)|$.
3. Determine a bound (a *tight* bound if possible) for $|f^{(n+1)}(x)| \le M$ on the interval, this will involve the symbol a.
4. In Taylor's inequality $\frac{M}{(n+1)!}|x-b|^{n+1}$ replace M and replace x by an endpoint (this will involve the symbol a).'
5. Then solve for a to get the desired error.

Given an interval $[b-a, b+a]$, find n so that $T_n(x)$ gives a desired error:
(There is no good general way to solve for the answer in this case, you just use trial and error).

1. Find the error for $n = 1$, then $n = 2$, then $n = 3$, etc. Once you get an error less than the desired error, you stop.
2. If you spot a pattern in the errors, then use the pattern to solve for the first time the error will be less than the desired error.

Facts about Taylor Series

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \quad \text{, for all } x.$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots \quad \text{, for all } x.$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots \quad \text{, for all } x.$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad \text{, for } -1 < x < 1.$$

Substituting into series (examples):

$$e^{2x^3} = \sum_{k=0}^{\infty} \frac{1}{k!} 2^k x^{3k} = 1 + 2x^3 + \tfrac{2^2}{2!}x^6 + \tfrac{2^3}{3!}x^9 + \cdots \quad \text{, for all } x.$$

$$\sin(5x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} 5^{2k+1} x^{2k+1} = 5x - \tfrac{5^3}{3!}x^3 + \tfrac{5^5}{5!}x^5 + \cdots \quad \text{, for all } x.$$

$$\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} = 1 - \tfrac{1}{2!}x^4 + \tfrac{1}{4!}x^8 + \cdots \quad \text{, for all } x.$$

$$\frac{1}{1+3x} = \sum_{k=0}^{\infty} (-3)^k x^k = 1 - 3x + 3^2x^2 - 3^3x^3 + \cdots \quad \text{, for } -1 < -3x < 1.$$

Multiplying out (examples):

$$x^3 e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+3} = x^3 + x^4 + \tfrac{1}{2!}x^5 + \tfrac{1}{3!}x^6 + \cdots \quad \text{, for all } x.$$

$$\frac{x^2}{1+2x} = \sum_{k=0}^{\infty} (-2)^k x^{k+2} = x^2 - 2x^3 + 2^2x^4 - 2^3x^5 + \cdots \quad \text{, for } -1 < 2x < 1.$$

Integrating/Differentiating (examples):

$$-\ln(1-x) = \int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots \quad \text{, for } -1 < x < 1.$$

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots \quad \text{, for } -1 < x < 1.$$

$$\int e^{x^3} dx = C + \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(3k+1)} x^{3k+1} = C + x + \tfrac{1}{2!(4)}x^4 + \tfrac{1}{3!(7)}x^7 + \cdots \quad \text{, for all } x.$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} k x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{, for } -1 < x < 1$$