Closing Thu: 12.3, 12.4(1), 12.4(2)  
Closing Tue: 12.5(1), 12.5(2)

Please read my 12.3, 12.4 review sheets. 
And look at the 12.5 visuals before Fri. 

12.4 The Cross Product

We define the cross product, or vector product, for two 3-dimensional vectors, 

\( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) and 
\( \mathbf{b} = \langle b_1, b_2, b_3 \rangle \), 

by 

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
i & j & k \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
\end{vmatrix} = 
\]

\[
= \begin{vmatrix}
a_2 & a_3 \\
b_2 & b_3 \\
\end{vmatrix} i - \begin{vmatrix}
a_1 & a_3 \\
b_1 & b_3 \\
\end{vmatrix} j + \begin{vmatrix}
a_1 & a_2 \\
b_1 & b_2 \\
\end{vmatrix} k 
\]

\[
= (a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k} 
\]

Ex: \( \mathbf{a} = \langle 1, 2, 0 \rangle \) and \( \mathbf{b} = \langle -1, 3, 2 \rangle \)
You do: $a = \langle 1,3,-1 \rangle$, $b = \langle 2,1,5 \rangle$.

Compute $a \times b$.
Most important fact:
The vector $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. 
Note: If \( \mathbf{a} \) and \( \mathbf{b} \) are parallel to each other, then there are many vectors perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \).
So what happens to \( \mathbf{v} = \mathbf{a} \times \mathbf{b} \)?

Example: Give me any two vectors that are parallel and let’s see.
Right-hand rule
If the fingers of the right-hand curl from \( \mathbf{a} \) to \( \mathbf{b} \), then the thumb points in the direction of \( \mathbf{a} \times \mathbf{b} \).
The magnitude of \( \mathbf{a} \times \mathbf{b} \):

Through some algebra and using the dot product rules, it can be shown that

\[
|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)
\]

where \( \theta \) is the smallest angle between \( \mathbf{a} \) and \( \mathbf{b} \). (\( 0 \leq \theta \leq \pi \))

Note: \( |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta) \)

is the area of the parallelogram formed by \( \mathbf{a} \) and \( \mathbf{b} \)
12.5 Intro to Lines in 3D
To describe 3D lines we use parametric equations.

Here is a 2D example
Consider the 2D line: $y = 4x + 5$.

(a) Find a vector parallel to the line.
   Call it vector $\mathbf{v}$.
(b) Find a vector whose head touches some point on the line when drawn from the origin.
   Call it vector $\mathbf{r}_0$.
(c) We can reach all other points on the line by walking along $\mathbf{r}_0$, then adding scale multiples of $\mathbf{v}$. 
This same idea works to describe any line in 2- or 3-dimensions.

The equation for a line in 3D:
\[ \mathbf{v} = \langle a, b, c \rangle = \text{parallel to the line}. \]
\[ \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle = \text{position vector} \]
then all other points, \((x, y, z)\), satisfy
\[ \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle, \]
for some number \(t\).

The above form \((\mathbf{r} = \mathbf{r}_0 + t\mathbf{v})\) is called the vector form of the line.

We also can write this in parametric form as:
\[
\begin{align*}
  x &= x_0 + at, \\
  y &= y_0 + bt, \\
  z &= z_0 + ct.
\end{align*}
\]

or in symmetric form:
\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]
Basic Example – Given Two Points:
Find parametric equations of the line thru \( P(3, 0, 2) \) and \( Q(-1, 2, 7) \).
General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

2. Two lines **intersect** if they have an \((x, y, z)\) point in common (use different parameters when you combine!)
   Note: The *acute angle of intersection* is the acute angle between the direction vectors.

3. Two lines are **skew** if they don’t intersect and aren’t parallel.