1. (12 pts) Evaluate

(a) $\int \frac{1}{\sqrt{x^2 - 6x + 13}}\, dx$

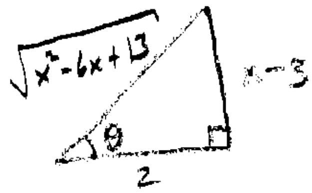
$$\underbrace{x^2 - 6x + 9}_{(x-3)^2} \underbrace{- 9 + 13}_{4}$$

$$\int \frac{1}{\sqrt{(x-3)^2 + 4}}\, dx$$

$$x - 3 = 2\tan\theta$$

$$dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{1}{2\sec\theta}\, 2\sec^2\theta\, d\theta$$



$$\int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{x^2-6x+13}}{2} + \frac{x-3}{2}\right| + C} = \ln\left|\sqrt{x^2-6x+13} + x - 3\right| + D$$

where

$D = -\ln(2) + C$

(b) $\int \frac{x^2 - 3x + 8}{x^2(x-2)}\, dx$

$$\frac{x^2 - 3x + 8}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$\Rightarrow \quad x^2 - 3x + 8 = Ax(x-2) + B(x-2) + Cx^2$$

$$x = 0 \Rightarrow 8 = B(-2) \Rightarrow B = -4$$

$$x = 2 \Rightarrow 4 - 6 + 8 = C(4) \Rightarrow 6 = 4C \Rightarrow C = 3/2$$

COEF. OF x^2: $\left.\begin{array}{l} \text{LHS} = 1 \\ \text{RHS} = A + C \end{array}\right\}$ $A + C = 1 \Rightarrow A = 1 - C = 1 - \frac{3}{2}$

$\Rightarrow A = -1/2$

$$\int \frac{-1/2}{x} + \frac{-4}{x^2} + \frac{3/2}{x-2}\, dx$$

$$= \boxed{-\frac{1}{2}\ln|x| + \frac{4}{x} + \frac{3}{2}\ln|x-2| + C}$$

2. (12 pts) Evaluate

(a) $\int_1^8 \frac{\ln(x)}{\sqrt[3]{x}}\,dx = \int_1^8 x^{-1/3}\ln(x)\,dx$

$u = \ln(x) \quad dv = x^{-1/3}dx$

$du = \frac{1}{x}dx \quad v = \frac{3}{2}x^{2/3}$

$= \frac{3}{2}x^{2/3}\ln(x)\Big|_1^8 - \int_1^8 \frac{3}{2}x^{-1/3}dx$

$= \frac{3}{2}\underbrace{(8)^{2/3}}_{4}\ln(8) - \left(\frac{9}{4}x^{2/3}\Big|_1^8\right)$

$= 6\ln(8) - \frac{9}{4}\left(\underbrace{8^{2/3}}_{4} - \underbrace{1^{2/3}}_{1}\right)$

$= \boxed{6\ln(8) - \frac{27}{4}} = 18\ln(2) - \frac{27}{4}$

≈ 5.7266

OR

$t = \sqrt[3]{x}$

$t^3 = x$

$3t^2dt = dx$

$\int_1^2 \frac{\ln(t^3)}{t}3t^2dt$

$= 3\int_1^2 t\ln(t^3)dt$

$= 9\int_1^2 t\ln(t)dt$

$\vdots$ BY PARTS

$= 9(\ln(4) - \frac{3}{4})$

$= 9\ln(4) - \frac{27}{4}$

$= 18\ln(2) - \frac{27}{4}$

(b) $\int \frac{\tan^3(x)}{\cos^4(x)}\,dx$

$= \int \tan^3(x)\sec^4(x)\,dx$

$= \int \tan^3(x)(\tan^2(x)+1)\sec^2(x)dx$

$u = \tan(x)$

$= \int u^3(u^2+1)du$

$= \int u^5 + u^3\,du$

$= \frac{1}{6}u^6 + \frac{1}{4}u^4 + C$

$= \boxed{\frac{1}{6}\tan^6(x) + \frac{1}{4}\tan^4(x) + C}$

OR

$\int (\sec^2(x)-1)\sec^3(x)\sec(x)\tan(x)dx$

$u = \sec(x)$

$\int (u^2-1)u^3\,du$

$= \int u^5 - u^3\,du$

$= \frac{1}{6}u^6 - \frac{1}{4}u^4 + c$

$= \boxed{\frac{1}{6}\sec^6(x) - \frac{1}{4}\sec^4(x) + C}$

EQUIVALENT

EQUIVALENT

OR $\int \frac{\sin^3(x)}{\cos^7(x)}dx = \int \frac{(1-\cos^2(x))}{\cos^7(x)}\sin(x)dx \quad u = \cos(x)$

$= \int \frac{1-u^2}{u^7}(-1)du = \int -u^{-7} + u^{-5}du = \frac{1}{6}u^{-6} - \frac{1}{4}u^{-4} + C$

$= \boxed{\frac{1}{6}\frac{1}{\cos^6(x)} - \frac{1}{4}\frac{1}{\cos^4(x)} + C}$

3. (12 pts) Evaluate

(a) $\int \cos(\sqrt{x})\, dx$

$t^2 = x$

$2t\,dt = dx$

$\int \cos(t)\, 2t\,dt$

$u = 2t \qquad dv = \cos(t)\,dt$

$du = 2\,dt \qquad v = \sin(t)$

$= 2t\sin(t) - \int 2\sin(t)\,dt$

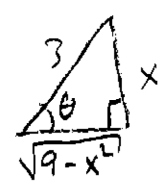
$= 2t\sin(t) + 2\cos(t) + C$

$= \boxed{2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C}$

(b) $\int \frac{x^2}{\sqrt{9-x^2}}\, dx$

$x = 3\sin(\theta)$

$dx = 3\cos(\theta)\, d\theta$



$\int \frac{9\sin^2(\theta)}{\sqrt{9\cos^2\theta}}\, 3\cos\theta\, d\theta$

$= \int 9\sin^2\theta\, d\theta = \frac{9}{2}\int 1 - \cos(2\theta)\, d\theta$

$= \frac{9}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$

$= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C$

$= \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C$

$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\,\frac{x}{3}\,\frac{\sqrt{9-x^2}}{3} + C$

$= \boxed{\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C}$

4. (12 pts) Answer the following questions:

(a) Find the average value of $f(x) = \dfrac{\sin(x)e^{\cos(x)}}{(e^{\cos(x)} + 1)^2}$ on the interval $x = 0$ to $x = \pi/2$.

$$\frac{1}{\frac{\pi}{2} - 0} \int_0^{\pi/2} \frac{\sin(x) e^{\cos(x)}}{(e^{\cos(x)} + 1)^2} dx$$

$u = e^{\cos(x)} + 1$

$du = -\sin(x) e^{\cos(x)} dx$

$$\frac{2}{\pi} \int_{e+1}^{2} \frac{(-1)}{u^2} du$$

$$\frac{2}{\pi} \left[\frac{1}{u} \Big|_{e+1}^{2} \right] = \boxed{\frac{2}{\pi} \left[\frac{1}{2} - \frac{1}{(e+1)} \right]} \approx 0.14710$$

(b) Evaluate the ***improper integral***: $\int_1^2 \frac{x}{\sqrt[4]{x-1}} dx$. (Give the value if it converges, or show why it diverges).

$$\lim_{t \to 1^+} \int_t^2 \frac{x}{(x-1)^{1/4}} dx$$

$u = x - 1$ $\quad x = u + 1$

$du = dx$

$$= \lim_{t \to 1^+} \left[\int_{t-1}^{1} \frac{u+1}{u^{1/4}} du \right]$$

$$= \lim_{t \to 1^+} \left[\int_{t-1}^{1} u^{3/4} + u^{-1/4} du \right]$$

$$= \lim_{t \to 1^+} \left[\frac{4}{7} u^{7/4} + \frac{4}{3} u^{3/4} \Big|_{t-1}^{1} \right]$$

CONVERGES

$$= \lim_{t \to 1^+} \left[\left(\frac{4}{7} + \frac{4}{3} \right) - \left(\cancel{\frac{4}{7}(t-1)^{7/4}} + \cancel{\frac{4}{3}(t-1)^{3/4}} \right) \right] = \frac{4}{7} + \frac{4}{3}$$

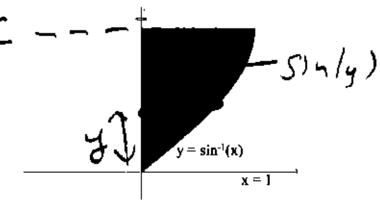
$$= \frac{12 + 28}{21} = \boxed{\frac{40}{21}} \approx 1.9048$$

5. (10 pts) Consider the region R in the first quadrant of the xy-plane bounded by $y = \sin^{-1}(x)$ and the y-axis (*the region is shaded in the picture below*). A water tank is formed by rotating this region about the y-axis. The tank starts full of water.

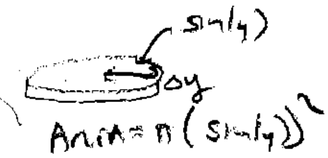
Assume all lengths are in meters. Recall the density of water is 1000 kg/m^3 and gravity is 9.8 m/s^2.

(a) Set up (DO NOT EVALUATE) an integral for the work required to pump all the water to the top of the tank.

FOR ANY $0 \le y \le \pi/2$, A HORIZONTAL SLICE WITH THICKNESS Δy WILL BE LIFTED A DIST $= \frac{\pi}{2} - y$ AND WILL WEIGH FORCE $= 9800 \cdot \pi\,(\sin(y))^2\,\Delta y$



$$\boxed{\text{WORK} = \int_0^{\pi/2} \left(\frac{\pi}{2} - y\right) 9800\pi \sin^2(y)\,dy}$$



(b) Use Simpson's rule with $n = 4$ to approximate your integral from part (a). Show some work in your calculations and give a final answer as a decimal accurate to 3 digits.

$$\Delta x = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}, \quad x_0 = 0,\ x_1 = \pi/8,\ x_2 = \pi/4,\ x_3 = 3\pi/8,\ x_4 = \pi/2$$

$$\frac{1}{3}\,\frac{\pi}{8} \cdot 9800\pi \left[\underbrace{\left(\frac{\pi}{2} - 0\right)(\sin(0))^2}_{0} + \underbrace{4\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\left(\sin\left(\frac{\pi}{8}\right)\right)^2}_{?} + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\sin^2\left(\frac{\pi}{4}\right) + 4\left(\frac{\pi}{2} - \frac{3\pi}{8}\right)\sin^2\left(\frac{3\pi}{8}\right) + \underbrace{\left(\frac{\pi}{2} - \frac{\pi}{2}\right)\sin^2\left(\frac{\pi}{2}\right)}_{0} \right]$$

$$\frac{\pi}{24}\, 9800\pi \left[0 + \underbrace{0.6901134 + 0.785398 + 1.340759}_{} + 0 \right]$$

$$\frac{9800\pi^2}{24} \left[2.8162701 \right] \approx \boxed{11349.818 \text{ JOULES}}$$

$$\approx 11350 \text{ Joules}$$

ASIDE

ACTUAL VALUE $\approx$ 11294 Joules