1. (14 points) Compute the following integrals.

(a) $\int_1^e 9\sqrt{x}\ln(x)\,dx$. BY PARTS

$u = \ln(x)$ $\quad dv = 9\sqrt{x}\,dx$

$du = \frac{1}{x}dx$ $\quad v = 6x^{3/2}$

$6x^{3/2}\ln(x)\Big|_1^e - \int_1^e 6x^{1/2}\,dx$

$[6(e)^{3/2}\ln(e) - 6(1)^{3/2}\ln(1)] - 4x^{3/2}\Big|_1^e$

$6e^{3/2} - 4e^{3/2} + 4$

$\boxed{2e^{3/2}+4}$ $\quad \approx 12.9634$

(b) $\int_0^1 \frac{x+1}{x^2+4}\,dx$ SEPARATE, SUBSTITUTION

$\int_0^1 \frac{x}{x^2+4}\,dx \quad + \quad \int_0^1 \frac{1}{x^2+4}\,dx$

$u = x^2+4$

$du = 2x\,dx$

$dx = \frac{1}{2x}du$

$\int_4^5 \frac{1}{u}\frac{1}{2}\,du \quad + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^1$

$\frac{1}{2}\ln|u|\Big|_4^5 \quad + \quad \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\tan^{-1}(0)$

$\boxed{\frac{1}{2}\ln(5) - \frac{1}{2}\ln(4) + \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{5}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)}$

≈ 0.343396

2. (14 points) Compute the following integrals.

(a) $\int \sqrt{16 - 6x - x^2}\, dx.$

COMPLETE THE SQUARE

$$\int \sqrt{25 - (x+3)^2}\, dx$$

$x+3 = 5\sin(\theta)$

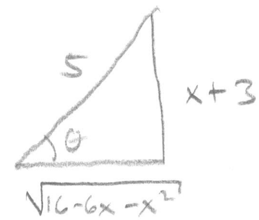
$dx = 5\cos(\theta)\,d\theta$

$$\int 5\cos(\theta)\, 5\cos(\theta)\, d\theta$$

$$25 \int \frac{1}{2}(1 + \cos(2\theta))\, d\theta$$

$$\frac{25}{2}\Big(\theta + \underbrace{\tfrac{1}{2}\sin(2\theta)}\Big) + C$$

$$\frac{25}{2}\big(\theta + \sin(\theta)\cos(\theta)\big) + C$$



$$\frac{25}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + \frac{25}{2}\,\frac{x+3}{5}\,\frac{\sqrt{16-6x-x^2}}{5} + C$$

$$\boxed{\frac{25}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + \frac{1}{2}(x+3)\sqrt{16-6x-x^2} + C}$$

(b) $\int \frac{x+1}{x^3 + 3x^2}\, dx\ .$

PARTIAL FRACTIONS

$$\frac{x+1}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

$$x+1 = Ax(x+3) + B(x+3) + Cx^2 = (A+C)x^2 + (3A+B)x + 3B$$

$x=0 \Rightarrow 1 = B\cdot 3 \Rightarrow \boxed{B = 1/3}$

$x=-3 \Rightarrow -2 = C\cdot 9 \Rightarrow \boxed{C = -2/9}$

$A+C=0 \Rightarrow \boxed{A = -C = 2/9}$

$$\int \frac{2/9}{x} + \frac{1/3}{x^2} + \frac{-2/9}{x+3}\, dx$$

$$= \boxed{\frac{2}{9}\ln|x| - \frac{1}{3x} - \frac{2}{9}\ln|x+3| + C = \frac{2}{9}\ln\left|\frac{x}{x+3}\right| - \frac{1}{3x} + C}$$

3. (8 pts) Consider the **improper** integral $\int_0^{\infty} \frac{x}{(x+a)^{5/2}}\,dx$, where a is a positive constant. If the integral converges, then find the value in terms of a. If it diverges, explain why.

$$\lim_{t\to\infty} \int_0^t \frac{x}{(x+a)^{5/2}}\,dx$$

$u = x + a$ $\qquad x = u - a$

$du = dx$

$$\lim_{t\to\infty} \int_a^{t+a} \frac{u-a}{u^{5/2}}\,du$$

$$\lim_{t\to\infty} \int_a^{t+a} u^{-3/2} - a u^{-5/2}\,du$$

$$\lim_{t\to\infty} \left[-2u^{-1/2} + \frac{2a}{3} u^{-3/2} \Big|_a^{t+a} \right]$$

$$\lim_{t\to\infty} \left[\left(-\cancel{\frac{2}{(t+a)^{1/2}}}^{\,0} + \frac{2a}{3(t+a)^{3/2}}^{\,0} \right) - \left(-\frac{2}{a^{1/2}} + \frac{2a}{3a^{3/2}} \right) \right]$$

$$\boxed{\begin{aligned} \frac{2}{a^{1/2}} - \frac{2}{3a^{1/2}} &= \frac{6}{3a^{1/2}} - \frac{2}{3a^{1/2}} \\ &= \frac{4}{3\sqrt{a}} \end{aligned}}$$

converges

4. (10 pts) Dr. Loveless goes for a jog on a straight path with velocity given by $v(t) = 3e^{-\sqrt{t}}$, where t is in hours and velocity is in miles per hour. At $t = 4$ hours of jogging, some former students jump out and throw water balloons at him. **Give units for all your answers below.**

(a) How far was Dr. Loveless from his starting at $t = 4$ hours?

$\int_0^4 3e^{-\sqrt{t}}\,dt$ $\qquad x = \sqrt{t} \qquad x^2 = t$

$2x\,dx = dt$

$\int_0^2 3e^{-x}\,2x\,dx$ $\qquad u = 6x \qquad dv = e^{-x}dx$

$du = 6dx \qquad v = -e^{-x}$

$-6xe^{-x}\Big|_0^2 - \int_0^2 -6e^{-x}\,dx$

$\left[-12e^{-2} - -6(0)e^{0}\right] + \left[-6e^{-x}\Big|_0^2\right]$

$-12e^{-2} + \left[-6e^{-2} - -6\right]$

$6 - 18e^{-2} = \boxed{6 - \frac{18}{e^2} \text{ miles}}$

≈ 3.56396 miles

(b) What was his average acceleration and average velocity over the first 4 hours?

i. Average velocity:

$\frac{1}{4-0}\int_0^4 v(t)\,dt = \boxed{\frac{1}{4}\left(6 - \frac{18}{e^2}\right) \frac{\text{miles}}{\text{hour}}}$

≈ 0.890991 mph

ii. Average acceleration:

$\frac{1}{4-0}\int_0^4 a(t)\,dt = \frac{1}{4}\left[v(4) - v(0)\right]$

$= \frac{1}{4}\left[3e^{-2} - 3e^{0}\right] = \boxed{\frac{3}{4}\left[e^{-2} - 1\right] \frac{\text{miles}}{\text{hr}^2}}$

$\approx -0.6484985 \frac{m}{hr^2}$

5. (8 pts) After Dr. Loveless dries off, he continues his work out. He starts to lift a sandbag. The sandbag weighs 50 pounds when it is on the ground. As he lifts the bag it leaks out sand at a constant linear rate. When the sandbag is lifted 2 feet, it weighs 46 pounds. Before he passes out, Dr. Loveless does 145 foot-pounds of work in lifting the sandbag. How high did he lift the sandbag?
(Hint: Start by finding the linear function for weight (force) in terms of height.)

$f(x) = mx + b \qquad m = \frac{46-50}{2-0} = -2$

$\boxed{f(x) = -2x + 50 \text{ pounds}}$ x = dist. from bottom.

x = 2 □ 46 lbs

x = 0 □ 50 lbs

only makes sense up to x = 25

$\text{WORK} = \int_0^a -2x + 50\,dx \overset{?}{=} 145 \text{ ft-lbs}$ FIND a = ?

$-x^2 + 50x \Big|_0^a$

$-a^2 + 50a \overset{?}{=} 145$

$0 = a^2 - 50a + 145$

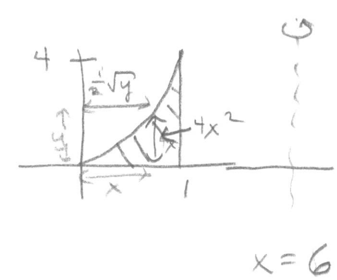
the other sol'n is 46.9089023

take the answer less than 25

$a = \frac{50 \pm \sqrt{50^2 - 4 \cdot 145}}{2}$

$\boxed{a = \frac{50 - \sqrt{50^2 - 4\cdot 145}}{2} = \frac{50 - \sqrt{1920}}{2} \approx 3.091977 \text{ feet}}$

6. (6 points) Consider the region, R, bounded by $y = 4x^2$ and the x-axis between $x = 0$ and $x = 1$. Using both methods, cylindrical shells and cross-sectional slicing, set up two integrals for the volume of the solid obtained by rotating the region R about the **vertical line** $x = 6$. Only set up, DO NOT EVALUATE.



(a) Cross-sectional slicing:

$$\boxed{\int_0^4 \pi\left(6 - \tfrac{1}{2}\sqrt{y}\right)^2 - \pi\, 5^2\, dy}$$

(b) Cylindrical Shells:

$= 14\pi \approx 43.9823$

$$\boxed{\int_0^1 2\pi(6 - x)\,4x^2\,dx}$$