1. (12 points) Evaluate

(a) $\int_0^4 \sqrt{x}e^{\sqrt{x}}\,dx$

$t^2 = x$

$2t\,dt = dx$

$= \int_0^2 t e^t \, 2t\,dt$

$u = 2t^2 \quad dv = e^t dt$

$du = 4t\,dt \quad v = e^t$

$= \int_0^2 2t^2 e^t\,dt$

$= 2t^2e^t\Big|_0^2 - \int_0^2 4te^t\,dt$

$u = 4t \quad dv = e^t dt$

$du = 4\,dt \quad v = e^t$

$= 8e^2 - \left[4te^t\Big|_0^2 - \int_0^2 4e^t dt\right]$

$= 8e^2 - \left[8e^2 - 4e^t\Big|_0^2\right]$

$= 4e^2 - 4e^0 = \boxed{4(e^2-1) = 4e^2-4} \approx 25.556$

(b) $\int \frac{x+9}{x^3+3x^2}\,dx$

$\frac{x+9}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$

$\Rightarrow x+9 = Ax(x+3) + B(x+3) + Cx^2$

$x=0 \Rightarrow 9 = 3B \Rightarrow B = 3$

$x=-3 \Rightarrow 6 = 9C \Rightarrow C = 2/3$

$A + C = 0 \Rightarrow A = -C = -2/3$

$\int \frac{-2/3}{x} + \frac{3}{x^2} + \frac{2/3}{x+3}\,dx$

$= \boxed{-\frac{2}{3}\ln|x| - \frac{3}{x} + \frac{2}{3}\ln|x+3| + C}$

2. (12 points) Evaluate

(a) $\displaystyle\int \frac{x}{(x^2+4x+13)^{3/2}}\,dx$

$x^2+4x+4-4+13$

$(x+2)^2+9$

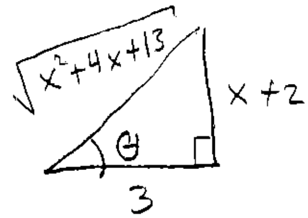
$x+2 = 3\tan\theta \Rightarrow (x+2)^2+9 = 9\sec^2\theta$

$x = 3\tan\theta - 2$

$dx = 3\sec^2\theta\, d\theta$

$$= \int \frac{(3\tan\theta - 2)}{(9\sec^2\theta)^{3/2}}\, 3\sec^2\theta\, d\theta$$

$$= \int \frac{(3\tan\theta - 2)}{27\sec^3\theta}\, 3\sec^2\theta\, d\theta$$



$$= \frac{1}{9}\int \frac{3\tan\theta}{\sec\theta} - \frac{2}{\sec\theta}\, d\theta$$

$$= \frac{1}{9}\int 3\sin\theta - 2\cos\theta\, d\theta$$

$$= \frac{1}{9}\left[-3\cos\theta - 2\sin\theta\right] + C$$

$$= \boxed{-\frac{1}{\cancel{3}}\,\frac{\cancel{3}}{\sqrt{x^2+4x+13}} - \frac{2}{9}\,\frac{x+2}{\sqrt{x^2+4x+13}} + C}$$

$$\rightarrow = \frac{-2x-13}{9\sqrt{x^2+4x+13}} + C$$

(b) $\displaystyle\int \frac{\cos(x)\sin(x)}{2\sqrt[3]{\sin(x)+2}}\,dx$

$t = \sin(x)+2 \qquad \sin(x) = t-2$

$dt = \cos(x)\,dx$

$\frac{1}{\cos(x)}\,dt = dx$

$$\int \frac{\cancel{\cos(x)}\,(t-2)}{2\sqrt[3]{t}}\,\frac{1}{\cos(x)}\,dt$$

$$\frac{1}{2}\int \frac{t}{t^{1/3}} - \frac{2}{t^{1/3}}\,dt$$

$$\frac{1}{2}\int t^{2/3} - 2t^{-1/3}\,dt$$

$$\frac{1}{2}\left[\frac{3}{5}t^{5/3} - 2\,\frac{3}{2}t^{2/3}\right] + C$$

$$\boxed{\frac{3}{10}(\sin(x)+2)^{5/3} - \frac{3}{2}(\sin(x)+2)^{2/3} + C}$$

3. (12 points) Evaluate

(a) $\int_0^{\pi/4} \cos^4(x)\,dx$ $= \int_0^{\pi/4} \cos^2(x)\cos^2(x)\,dx$

$$= \int_0^{\pi/4} \frac{1}{2}(1+\cos(2x))\frac{1}{2}(1+\cos(2x))\,dx$$

$$= \frac{1}{4}\int_0^{\pi/4} 1 + 2\cos(2x) + \cos^2(2x)\,dx$$

$$= \frac{1}{4}\int_0^{\pi/4} 1 + 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\,dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x)\Big|_0^{\pi/4}\right]$$

$$= \frac{1}{4}\left[\left(\frac{3\pi}{8} + \sin(\pi/2) + 0\right) - (0)\right]$$

$$= \boxed{\frac{3\pi}{32} + \frac{1}{4}} \approx 0.54452$$

(b) $\int \frac{x^4+x+1}{x^2+9}\,dx$

$$\begin{array}{r} x^2 - 9 \\ x^2+9 \,\big)\, \overline{x^4 + x + 1} \\ -(x^4 + 9x^2) \\ \hline -9x^2 + x + 1 \\ -(-9x^2 - 81) \\ \hline x + 82 \end{array}$$

$$= \int x^2 - 9 + \frac{x+82}{x^2+9}\,dx$$

$$= \frac{1}{3}x^3 - 9x + \int \frac{x}{x^2+9}\,dx + \int \frac{82}{x^2+9}\,dx$$

$$= \boxed{\frac{1}{3}x^3 - 9x + \frac{1}{2}\ln|x^2+9| + \frac{82}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

4. (12 pts)

(a) Use the trapezoid rule with $n = 4$ to approximate the arc length of $f(x) = x^2 - x$ from $x = 0$ to $x = 2$. (You can leave your answer expanded out with all the correct numbers in all the correct places)

$f'(x) = 2x - 1$, $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, $x_4 = 2$

$$\int_0^2 \sqrt{1 + (2x-1)^2}\, dx$$

$$\approx \boxed{\frac{1}{2}\left(\frac{1}{2}\right)\left[\sqrt{1+(-1)^2} + 2\sqrt{1+(0)^2} + 2\sqrt{1+(1)^2} + 2\sqrt{1+2^2} + \sqrt{1+3^2}\right]}$$

(b) Consider the improper integral $\int_1^\infty \frac{\ln(x)}{\sqrt{x}}\, dx$. Determine if it converges or diverges. If it converges give the value. (You MUST write as a limit, integrate and show your work).

$$\lim_{t\to\infty} \int_1^t \frac{\ln(x)}{\sqrt{x}}\, dx$$

$u = \ln(x)$ $dv = x^{-1/2}\, dx$

$du = \frac{1}{x}dx$ $v = 2x^{1/2}$

$$= \lim_{t\to\infty}\left[2\sqrt{x}\ln(x)\Big|_1^t - \int_1^t 2x^{-1/2}\, dx\right]$$

$$= \lim_{t\to\infty}\left[2\sqrt{t}\ln(t) - \left(4x^{1/2}\Big|_1^t\right)\right]$$

$$= \lim_{t\to\infty}\left[2\sqrt{t}\ln(t) - (4\sqrt{t} - 4)\right]$$

$$= \lim_{t\to\infty}\left[2\sqrt{t}\ln(t) - 4\sqrt{t} + 4\right]$$

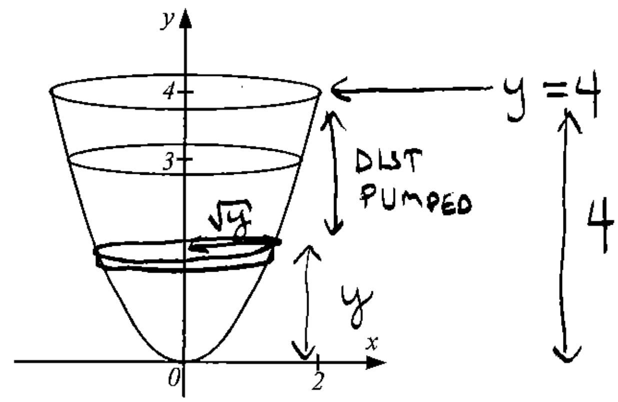
$$= \lim_{t\to\infty}\left[\sqrt{t}\,(2\ln(t) - 4) + 4\right]$$

∞

$\boxed{\text{DIVERGES}}$

5. (12 points) The portion of the graph $y = x^2$ between $x = 0$ and $x = 2$ is rotated around the y-axis to form a container. The container is partially filled with water, up to the level $y = 3$.

Find the work required to pump all of the water out over the top of the side of the container. Give your answer (in joules) in exact form. (Distance is in meters, the density is 1000 kg/m^3, and the acceleration due to gravity is 9.8 m/s^2.)



CONSIDER A THIN HORIZONTAL SLICE OF WATER AT HEIGHT y WITH THICKNESS Δy

DISTANCE PUMPED $= 4 - y$

FORCE/WEIGHT OF SLICE $= 9800\pi(\sqrt{y})^2 \Delta y$

$$\boxed{\int_0^3 9800\pi\, y\,(4-y)\,dy}$$

$$9800\pi \int_0^3 4y - y^2\, dy$$

$$9800\pi \left(2y^2 - \tfrac{1}{3}y^3 \Big|_0^3\right)$$

$$9800\pi\left[(18-9)-0\right]$$

$$= \boxed{88{,}200\pi \text{ JOULES}} \approx 277088.47205 \text{ JOULES}$$