1. (11 pts) Evaluate the integrals.

(a) $\displaystyle\int \frac{x - 4\sqrt[3]{x}}{x^2} + \frac{5}{2\sqrt{x}}\,dx$

$$= \int \frac{x}{x^2} - \frac{4x^{1/3}}{x^2} + \frac{5}{2}x^{-1/2}\,dx$$

$$= \int \frac{1}{x} - 4x^{-5/3} + \frac{5}{2}x^{-1/2}\,dx$$

$$= \ln|x| - 4\cdot\frac{-3}{2}x^{-2/3} + \frac{5}{2}(2)x^{\frac{1}{2}} + C$$

$$= \boxed{\ln|x| + 6x^{-2/3} + 5x^{\frac{1}{2}} + C}$$

(b) $\displaystyle\int_{\sqrt{2}}^{2} x^3\left(\frac{1}{2}x^2 - 1\right)^4 dx$

$u = \frac{1}{2}x^2 - 1 \quad \leftrightarrow \quad x^2 = 2(u+1)$

$du = x\,dx \quad \leftrightarrow \quad \frac{1}{x}\,du = dx$

$$= \int_0^1 x^{\cancel{3}\,2} u^4 \frac{1}{\cancel{x}}\,du$$

$$= \int_0^1 2(u+1)u^4\,du$$

$$= 2\int_0^1 u^5 + u^4\,du$$

$$= 2\left(\frac{1}{6}u^6 + \frac{1}{5}u^5\right)\Big|_0^1$$

$$= 2\left[\left(\frac{1}{6} + \frac{1}{5}\right) - (0)\right] = \frac{1}{3} + \frac{2}{5} = \frac{5+6}{15} = \boxed{\frac{11}{15}}$$

2. (11 pts) Evaluate the integrals.

(a) $\displaystyle\int \csc^2(x) + 4^x + \frac{\ln(x)}{x}\,dx$

$$= \int \csc^2(x)dx + \int 4^x dx + \int \frac{\ln(x)}{x} dx$$

$u = \ln(x)$

$du = \frac{1}{x} dx$

$$= -\cot(x) + \frac{1}{\ln(4)} 4^x + \int u\,du$$

$$= -\cot(x) + \frac{1}{\ln(4)} 4^x + \frac{1}{2} u^2 + C$$

$$= \boxed{-\cot(x) + \frac{1}{\ln(4)} 4^x + \frac{1}{2}(\ln(x))^2 + C}$$

(b) $\displaystyle\int_0^{\pi/4} \frac{\sin(2x)}{(\cos(2x)+1)^3}\,dx$

$u = \cos(2x) + 1$

$du = -2\sin(2x)\,dx$

$\frac{1}{-2\sin(2x)} du = 1\,dx$

$$= \int_2^1 \frac{\cancel{\sin(2x)}}{u^3} \frac{1}{-2\cancel{\sin(2x)}} du$$

$$= -\frac{1}{2}\int_2^1 u^{-3}\,du = \frac{1}{2}\int_1^2 u^{-3} du$$

$$= \frac{1}{2} \frac{1}{-2} u^{-2}\Big|_1^2$$

$$= -\frac{1}{4} \frac{1}{u^2}\Big|_1^2$$

$$= -\frac{1}{4}\left(\frac{1}{2^2} - \frac{1}{1^2}\right)$$

$$= -\frac{1}{4}\left(\frac{1}{4} - 1\right) = \boxed{\frac{3}{16}}$$

3. (a) (5 pts) Consider $\lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + \left(\frac{2i}{n}\right)^3\right) \frac{2}{n}$. Rewrite this as an integral and evaluate.

$$\Delta x = \frac{b-a}{n} = \frac{2}{n} \Rightarrow b-a=2$$

$$x_i = a + i\Delta x = a + \frac{2i}{n} = 0 + \frac{2i}{n} \Rightarrow a=0 \Rightarrow b=2$$

$$\int_0^2 5 + x^3\,dx = 5x + \frac{1}{4}x^4 \Big|_0^2$$
$$= \left(10 + \frac{1}{4}2^4\right) - 0$$
$$= 10 + 4 = \boxed{14}$$

ASIDE: $a=5$ IS NOT CORRECT!

$x_i = a + \frac{2i}{n}$ AND THE EXPRESSION ABOVE DOES NOT LOOK LIKE $\left(5 + \frac{2i}{n}\right)^3$ ← in this case $a$ would be 5

(b) (8 pts) Consider the function $f(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}t^2} dt$. Do NOT try to integrate, you can't! (Aside: This is a very important function in probability and statistics).

i. Find the equation for the tangent line to $f(x)$ at $x=0$. (Hint: First, find $f'(x)$).

$$f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \Rightarrow f'(0) = \frac{1}{\sqrt{2\pi}}$$

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_0^0 e^{-\frac{1}{2}t^2} dt = 0$$

$$\underbrace{y = f(0) + f'(0)(x-0)}_{\text{THIS IS AN EQUATION FOR A TANGENT LINE!}} \Rightarrow \boxed{y = \frac{1}{\sqrt{2\pi}} x}$$

ii. Estimate the value of $f(1) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-\frac{1}{2}t^2} dt$ using the midpoint rule method and $n=3$ subdivisions. You do not have to simplify your answer, just show me the expanded answer with all the correct numbers in the correct places.

$$\Delta x = \frac{1-0}{3} = \frac{1}{3}, \quad x_0 = 0, \; x_1 = \frac{1}{3}, \; x_2 = \frac{2}{3}, \; x_3 = 1$$
$$\bar{x}_1 = \frac{1}{6}, \; \bar{x}_2 = \frac{1}{2}, \; \bar{x}_3 = \frac{5}{6}$$

$$\boxed{f(1) \approx \frac{1}{\sqrt{2\pi}} \left[ e^{-\frac{1}{2}\left(\frac{1}{6}\right)^2} \cdot \frac{1}{3} + e^{-\frac{1}{2}\left(\frac{1}{2}\right)^2} \cdot \frac{1}{3} + e^{-\frac{1}{2}\left(\frac{5}{6}\right)^2} \cdot \frac{1}{3} \right]}$$

4. (12 pts) The two parts below are not related [redacted]

(a) The velocity of an object moving along a straight line is given by $v(t) = 10\sin\left(\frac{\pi}{2}t\right)$ miles/hour. Find the **total distance** traveled by the object from $t = 0$ to $t = 3$ hours.

$$\int_0^3 \left|10\sin\left(\tfrac{\pi}{2}t\right)\right| dt = ?$$

$10\sin\left(\frac{\pi}{2}t\right) \overset{?}{=} 0$

$\frac{\pi}{2}t = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$

$t = \ldots, -2, 0, \boxed{2}, 4, 6, \ldots$

only value between 0 and 3 where velocity changes sign

[1] $\int_0^2 10\sin\left(\frac{\pi}{2}t\right)dt$

$u = \frac{\pi}{2}t$

$du = \frac{\pi}{2}dt$

$dt = \frac{2}{\pi}du$

$= \int_0^\pi 10\sin(u)\frac{2}{\pi}du$

$= \frac{20}{\pi}(-\cos(u))\Big|_0^\pi = \frac{20}{\pi}[(-(-1)) - (-1)] = \frac{40}{\pi}$ miles

[2] $\int_2^3 10\sin\left(\frac{\pi}{2}t\right)dt = \int_\pi^{3\pi/2} 10\sin(u)\frac{2}{\pi}du = \frac{20}{\pi}(-\cos(u))\Big|_\pi^{3\pi/2} = \frac{20}{\pi}[(-0) - -(-1)] = -\frac{20}{\pi}$

$$\boxed{\text{TOTAL DISTANCE} = \frac{40}{\pi} + \frac{20}{\pi} = \frac{60}{\pi} \text{ miles}}$$

(b) At time $t = 0$ seconds, a small water balloon is thrown downward from the top of a tall building toward the ground (where your math instructor happens to be sitting). At $t = \frac{1}{2}$ second, the balloon is 90 feet above the ground. At $t = 2$ seconds, the balloon hits the ground. Assume acceleration is a constant 32 feet/second$^2$ downward. At what velocity does the water balloon hit the ground?

$a(t) = -32$

$v(t) = -32t + C$

$h(t) = -16t^2 + Ct + D$

$h\left(\frac{1}{2}\right) = 90 \Rightarrow -16\left(\frac{1}{2}\right)^2 + \frac{1}{2}C + D = 90 \Rightarrow -4 + \frac{1}{2}C + D = 90 \Rightarrow \frac{1}{2}C + D = 94$

$h(2) = 0 \Rightarrow -16(2)^2 + 2C + D = 0 \Rightarrow -64 + 2C + D = 0 \Rightarrow 2C + D = 64$

$-\frac{3}{2}C + 0 = 30$

$C = -20$

initial velocity

$v(t) = -32t - 20$

$v(2) = -32(2) - 20 = -64 - 20 = \boxed{-84 \text{ ft/sec}}$

velocity when it hits ground

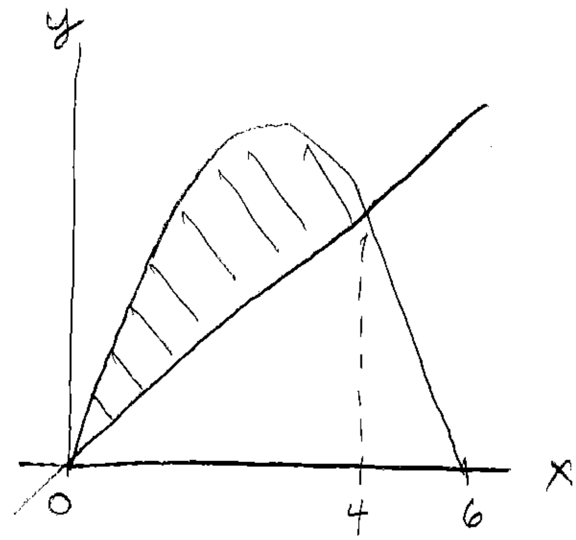
5. (13 pts) Consider the region $R$ bounded by $y = 6x - x^2$ and $y = 2x$.

(a) Draw the region $R$ and find the area.

$y = 6x - x^2 = x(6-x)$

$6x - x^2 \stackrel{?}{=} 2x \Rightarrow 0 = x^2 - 4x$

$0 = x(x-4)$




$\int_0^4 6x - x^2 - 2x\,dx$

$= \int_0^4 4x - x^2\,dx$

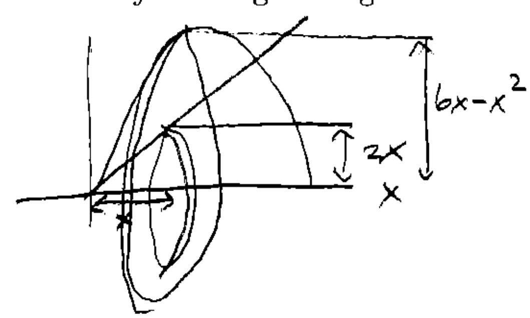
$= 2x^2 - \frac{1}{3}x^3 \Big|_0^4$

$= \left(2(4)^2 - \frac{1}{3}(4)^3\right) - 0$

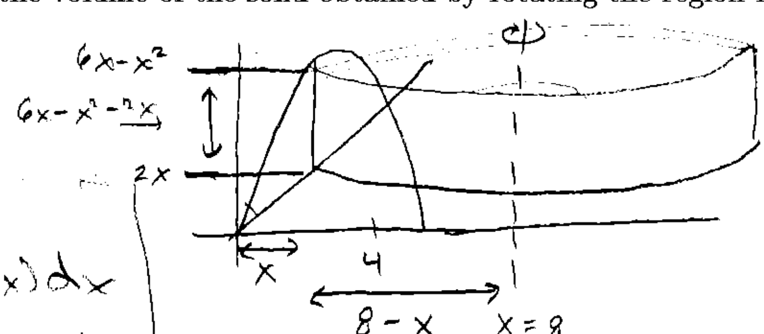
$= 32 - \frac{64}{3} = \frac{96-64}{3} = \boxed{\frac{32}{3}}$ units$^2$

(b) Set up an integral that represents the volume of the solid obtained by rotating the region $R$ about the $x$-axis.
(DO NOT EVALUATE)

$$\boxed{\int_0^4 \pi(6x-x^2)^2 - \pi(2x)^2\,dx}$$




(c) Set up an integral that represents the volume of the solid obtained by rotating the region $R$ about the **vertical** line $x = 8$.
(DO NOT EVALUATE)




$$\int_0^4 2\pi(8-x)(6x-x^2-2x)\,dx$$
$$= \int_0^4 2\pi(8-x)(4x-x^2)\,dx$$