7.2/7.3: ESSENTIAL TRIG IDENTITIES

We can get all the identities we need for this course from the four identities (Formulas (2), (3), and (4) can be derived from the law of cosines, see exercises 83, 85 and 86 in Appendix D if you want to learn more).

(1) $\sin^2(x) + \cos^2(x) = 1$, $\cos^2(x) = 1 - \sin^2(x)$, $\sin^2(x) = 1 - \cos^2(x)$

(2) $\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

(3) $\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

(4) $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

Here's how to get the other identities we need:

Dividing (1) by $\cos^2(x)$ gives: $\tan^2(x) + 1 = \sec^2(x)$, $\tan^2(x) = \sec^2(x) - 1$

Dividing (1) by $\sin^2(x)$ gives: $1 + \cot^2(x) = \csc^2(x)$

Plugging $A = B = x$ into (2) gives: $\sin(x)\cos(x) = \frac{1}{2}(\sin(2x))$

Plugging $A = B = x$ into (3) gives: $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

Plugging $A = B = x$ into (4) gives: $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

Plugging $A = mx$ and $B = nx$ into (2), (3) and (4) give

$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin((m-n)x) + \sin((m+n)x)]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m-n)x) + \cos((m+n)x)]$$
$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]$$

7.3: Trig Substitution - Intro

1. If integral involves $\sqrt{a^2 - x^2}$, then use $x = a\sin(\theta)$.
2. If integral involves $\sqrt{a^2 + x^2}$, then use $x = a\tan(\theta)$.
3. If integral involves $\sqrt{x^2 - a^2}$, then use $x = a\sec(\theta)$.

Examples: This just shows how to start problems in 7.3. We also need to talk about how to end problems and some subtleties about restrictions on θ.

1. $\int x^2\sqrt{4 - x^2}\,dx$. Use $u = 2\sin(\theta)$, $du = 2\cos(\theta)d\theta$.
   Then $\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2(\theta)} = \sqrt{4(1 - \sin^2(\theta))} = \sqrt{4\cos^2(\theta)} = 2\cos(\theta)$.
   Thus, $\int x^2\sqrt{4 - x^2}\,dx = \int 4\sin^2(\theta)2\cos(\theta)2\cos(\theta)d\theta = 16\int \sin^2(\theta)\cos^2(\theta)\,d\theta$. Now use 7.2 methods.

2. $\int \frac{x^3}{\sqrt{9 + x^2}}\,dx$. Use $u = 3\tan(\theta)$, $du = 3\sec^2(\theta)d\theta$.
   Then $\sqrt{9 + x^2} = \sqrt{9 + 9\tan^2(\theta)} = \sqrt{9(1 + \tan^2(\theta))} = \sqrt{9\sec^2(\theta)} = 3\sec(\theta)$.
   Thus, $\int \frac{x^3}{\sqrt{9 + x^2}}\,dx = \int \frac{9\tan^3(x)}{3\sec(\theta)}3\sec^2(\theta)d\theta = 9\int \tan^3(\theta)\sec(\theta)\,d\theta$. Now use 7.2 methods.

3. $\int \frac{\sqrt{x^2 - 25}}{x}\,dx$. Use $u = 5\sec(\theta)$, $du = 5\sec(\theta)\tan(\theta)d\theta$.
   Then $\sqrt{x^2 - 25} = \sqrt{25\sec^2(\theta) - 25} = \sqrt{25(\sec^2(\theta) - 1)} = \sqrt{25\tan^2(\theta)} = 5\tan(\theta)$.
   Thus, $\int \frac{\sqrt{x^2 - 25}}{x}\,dx = \int \frac{5\tan(\theta)}{5\sec(\theta)}5\sec(\theta)\tan(\theta)d\theta = 5\int \tan^2(\theta)\,d\theta$. Now use 7.2 methods.