1. (12 points) Compute the following integrals.

(a) $\displaystyle\int_0^2 \frac{x^2+3x-5}{x+1}\,dx.$

divide or substitute $\leftarrow u = x+1$

$$\begin{array}{r} x+2 \\ x+1 \,\overline{)\, x^2+3x-5} \\ -(x^2+x) \\ \hline 2x-5 \\ -(2x+2) \\ \hline -7 \end{array}$$

$$\int_0^2 x+2-\frac{7}{x+1}\,dx$$

$$\frac{1}{2}x^2+2x-7\ln|x+1|\Big|_0^2$$

$$(2+4-7\ln(3))-(0+0-7\ln(1))$$

$$\boxed{6-7\ln(3)} \approx -1.69029$$

(b) $\displaystyle\int x^4\ln(3x)+e^{-5x}\,dx$

separate, then by parts

$$\int e^{-5x}\,dx = -\frac{1}{5}e^{-5x}+C$$

$$\int x^4\ln(3x)\,dx$$

$u=\ln(3x)$ $\quad dv = x^4\,dx$

$du = \frac{3}{3x}\,dx$ $\quad v=\frac{1}{5}x^5$

$$\frac{1}{5}x^5\ln(3x)-\int\frac{1}{5}x^4\,dx$$

$$\boxed{\frac{1}{5}x^5\ln(3x)-\frac{1}{25}x^5-\frac{1}{5}e^{-5x}+C}$$

2. (12 points) Compute the following integrals.

(a) $\displaystyle\int \frac{x^2+7}{x^2(3-x)}\,dx.$ partial fractions

$$\frac{x^2+7}{x^2(3-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3-x}$$

$$x^2+7 = Ax(3-x) + B(3-x) + Cx^2 = (-A+C)x^2 + (3A-B)x + 3B$$

$x=0 \Rightarrow 7 = 3B \Rightarrow B = 7/3$

$x=3 \Rightarrow 16 = 9C \Rightarrow C = 16/9$

$-A+C=1 \Rightarrow A = C-1 = \frac{16}{9} - 1 = 7/9$

$$\int \frac{7/9}{x} + \frac{7/3}{x^2} + \frac{16/9}{3-x}\,dx$$

$$\boxed{\frac{7}{9}\ln|x| - \frac{7}{3x} - \frac{16}{9}\ln|3-x| + C}$$

(b) $\displaystyle\int \frac{1}{(x^2+6x+13)^{3/2}}\,dx.$ trig. substitution

$$x^2+6x+13 = x^2+6x+9-9+13 = (x+3)^2+4$$

$x+3 = 2\tan(\theta)$

$dx = 2\sec^2(\theta)\,d\theta$

$$\int \frac{1}{(4\tan^2(\theta)+4)^{3/2}}\, 2\sec^2(\theta)\,d\theta$$

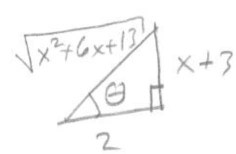
$$\int \frac{2\sec^2(\theta)}{(4\sec^2(\theta))^{3/2}}\,d\theta = \frac{2}{8}\int \frac{\sec^2(\theta)}{\sec^3(\theta)}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec(\theta)}\,d\theta$$

$$= \frac{1}{4}\int \cos(\theta)\,d\theta$$

$$= \frac{1}{4}\sin(\theta) + C$$

$$= \boxed{\frac{1}{4}\,\frac{x+3}{\sqrt{x^2+6x+13}} + C}$$



3. (12 points) Answer the following questions

(a) (6 pts) Find the average value of $f(x) = \sin^3(2x)\cos^5(2x)$ on the interval $x = \frac{\pi}{4}$ to $\cancel{\frac{\pi}{4}}$. $x = \pi/2$

$$\frac{1}{\frac{\pi}{2} - \pi/4} \int_{\pi/4}^{\pi/2} \sin^3(2x)\cos^5(2x)\,dx$$

$u = \cos(2x)$

$du = -2\sin(2x)\,dx$

$dx = -\frac{1}{2\sin(2x)}\,du$

$$\frac{4}{\pi} \int_{\pi/4}^{\pi/2} (1-\cos^2(2x))\cos^5(2x)\sin(2x)\,dx$$

$$\frac{4}{\pi} \int_{0}^{-1} (1-u^2)u^5 \frac{-1}{2}\,du$$

$$\frac{2}{\pi} \int_{-1}^{0} u^5 - u^7\,du$$

$$\frac{2}{\pi}\left(\frac{1}{6}u^6 - \frac{1}{8}u^8 \Big|_{-1}^{0}\right) = \boxed{\frac{2}{\pi}\left(\frac{-1}{6} + \frac{1}{8}\right) = \frac{1}{\pi}\left(\frac{-1}{3} + \frac{1}{4}\right) = \frac{-1}{12\pi}}$$

≈ -0.0265

(b) (6 pts) Evaluate: $\displaystyle\int_1^\infty \frac{1}{(x+1)\sqrt{x}}dx$. (Give the value if it converges, or show why it diverges).

$u = \sqrt{x} \Rightarrow u^2 = x$

$2u\,du = dx$

$$\lim_{t\to\infty} \int_1^t \frac{1}{(x+1)\sqrt{x}}\,dx$$

$$\lim_{t\to\infty} \int_1^{\sqrt{t}} \frac{1}{(u^2+1)\cancel{u}}\,2u\,du$$

$$= \lim_{t\to\infty} 2\int_1^{\sqrt{t}} \frac{1}{u^2+1}\,du$$

$$= \lim_{t\to\infty} 2\left(\tan^{-1}(u)\Big|_1^{\sqrt{t}}\right)$$

$$= \lim_{t\to\infty} 2\left(\tan^{-1}(\sqrt{t}) - \tan^{-1}(1)\right)$$

$$= 2\left(\pi/2 - \pi/4\right) = 2\pi/4 = \boxed{\pi/2} \approx 1.5708$$

4. (12 pts)

(a) (6 pts) Find the length of the curve $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ from $x = 1$ to $x = 5$.

$$\int_1^5 \sqrt{1 + \left(\frac{1}{2}x - \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^5 \sqrt{1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}}\, dx$$

$$= \int_1^5 \sqrt{\frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2}}\, dx$$

$$= \int_1^5 \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^5 \frac{1}{2}x + \frac{1}{2x}\, dx$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\ln|x|\Big|_1^5 = \left(\frac{25}{4} + \frac{1}{2}\ln(|5|)\right) - \left(\frac{1}{4} + 0\right)$$

$$= \frac{24}{4} + \frac{1}{2}\ln(5) = \boxed{6 + \frac{1}{2}\ln(5)}$$

$$\approx 6.8047$$

(b) (6 pts) Approximate $\int_1^3 \sqrt{x^3+1}\, dx$ using Simpson's rule and $n = 4$ subdivisions. (Give your final answer as a decimal to four digits).

$$\Delta x = \frac{3-1}{4} = \frac{1}{2} \qquad x_0 = 1,\ x_1 = \frac{3}{2},\ x_2 = 2,\ x_3 = \frac{5}{2},\ x_4 = 3$$

$$\frac{1}{3}\cdot\frac{1}{2}\left[\sqrt{1^3+1} + 4\sqrt{\left(\tfrac{3}{2}\right)^3+1} + 2\sqrt{2^3+1} + 4\sqrt{\left(\tfrac{5}{2}\right)^3+1} + \sqrt{3^3+1}\right]$$

$$\frac{1}{6}\left[\sqrt{2} + 4\sqrt{\tfrac{35}{8}} + 2\sqrt{9} + 4\sqrt{\tfrac{133}{8}} + \sqrt{28}\right]$$

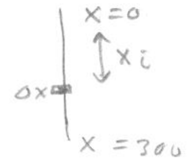
$$\approx \frac{1}{6}\cdot 37.38182288 \approx 6.2303038$$

$$6.2303$$

ACTUAL VALUE ≈ 6.229959387888

5. (12 points) A large crane is 300 feet above the ground. It has a cable with density 5 lbs/ft that reaches all the way down to the ground.

(a) If a 100 lbs object is attached to the bottom of the cable, how much total work is done in lifting the object the entire 300 feet?



LIFT OBJECT $= \underbrace{100}_{\text{CONST. FORCE}} \cdot \underbrace{300}_{\text{DIST}} = 30{,}000$ ft-lbs

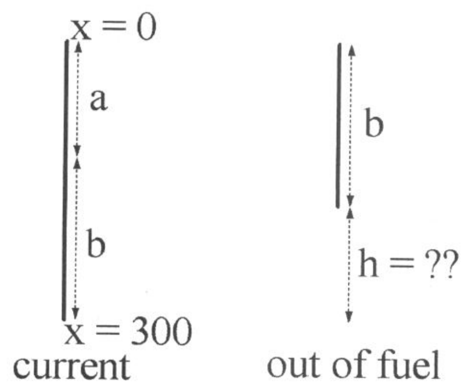
LIFT CABLE $\approx \sum_{i=1}^{n} 5\Delta x\, x_i$

$= \int_0^{300} 5x\,dx = \frac{5}{2}x^2\Big|_0^{300} = \frac{5}{2}\,90000 = 225{,}000$ ft-lbs

TOTAL $= 30{,}000 + 225{,}000 = \boxed{255{,}000 \text{ ft-lbs}}$

(b) At the end of the day, there is only enough fuel left for the crane to do 20,000 ft-lbs of work. Currently, the cable is extended the full 300 feet to the ground and there is NO object attached to the end. How high can it lift the cable before it runs out of fuel?

A picture is provided with labels you should find helpful, I suggest you find a first. (Give your final answer for h as a decimal to four digits).



WORK TO LIFT $x=0$ TO $x=a$ SEGMENT $= \int_0^a 5x\,dx$

$= \frac{5}{2}x^2\Big|_0^a = \frac{5}{2}a^2$

WORK TO LIFT $x=a$ to $x=300$ SEGMENT $= \underbrace{5b}_{\text{CONST. FORCE}} \cdot \underbrace{a}_{\text{DIST}} = 5ab$

WANT: $\frac{5}{2}a^2 + 5ab = 20{,}000$

$b = 300 - a \Rightarrow \frac{5}{2}a^2 + 5a(300-a) = 20000$

$\frac{5}{2}a^2 - 5a^2 + 1500a - 20000 = 0$

$-\frac{5}{2}a^2 + 1500a - 20000 = 0$

$-5a^2 + 3000a - 40000 = 0$

$a = \dfrac{-3000 \pm \sqrt{3000^2 - 4(-5)(-40000)}}{2(-5)}$

$a = \dfrac{-3000 \pm \sqrt{8200000}}{-10}$

$a = 13.64357$ or ~~$a = 586.3564$~~

SAME

Thus, $b = 300 - a = 286.35642$

$\boxed{h = 13.6436 \text{ ft}}$