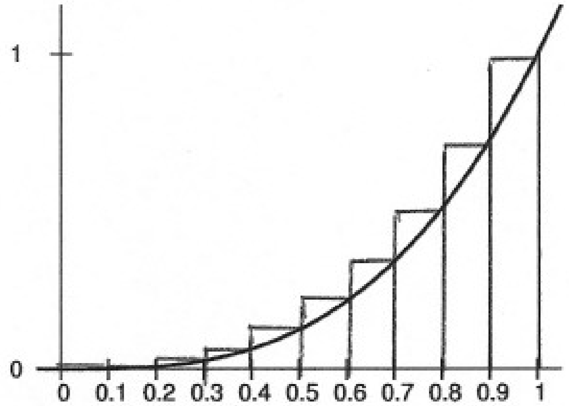
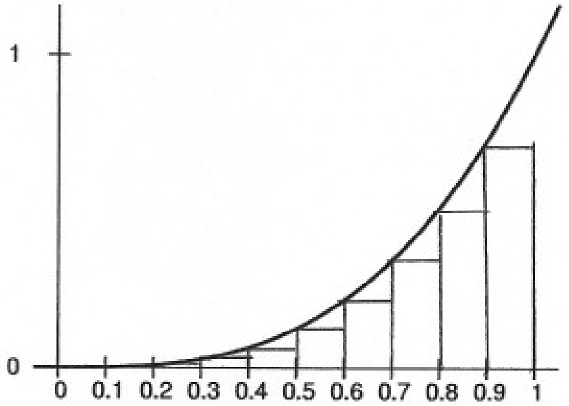
Estimating the Area with 10 Rectangles and Right Endpoints



$$R_{10} = S_1 + S_2 + \cdots + S_9 + S_{10}$$
$$R_{10} = w_1h_1 + w_2h_2 + \cdots + w_9h_9 + w_{10}h_{10}$$
$$R_{10} = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_9)\Delta x + f(x_{10})\Delta x$$
$$R_{10} = \sum_{i=1}^{10} f(x_i)\Delta x = \frac{1}{10}\left(\frac{1}{10}\right)^3 + \frac{1}{10}\left(\frac{2}{10}\right)^3 + \cdots + \frac{1}{10}\left(\frac{10}{10}\right)^3$$
$$R_{10} = 0.3025$$

Estimating the Area with 10 Rectangles and Left Endpoints



$$L_{10} = S_1 + S_2 + \cdots + S_9 + S_{10}$$
$$L_{10} = w_1h_1 + w_2h_2 + \cdots + w_9h_9 + w_{10}h_{10}$$
$$L_{10} = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_8)\Delta x + f(x_9)\Delta x$$
$$L_{10} = \sum_{i=1}^{10} f(x_{i-1})\Delta x = \frac{1}{10}\left(\frac{0}{10}\right)^3 + \frac{1}{10}\left(\frac{1}{10}\right)^3 + \frac{1}{10}\left(\frac{2}{10}\right)^3 + \cdots + \frac{1}{10}\left(\frac{9}{10}\right)^3$$
$$L_{10} = 0.2025$$

Summary of Finding the Area under $y = x^3$ from 0 to 1

n	R_n	L_n
4	0.390625	0.140625
5	0.36	0.16
10	0.3025	0.2025
100	0.255025	0.245025
1000	0.25050025	0.24950025
10000	0.2499500025	0.2500500025

For our example, $f(x) = x^3$ and we are finding the area under the graph from $a = 0$ and $b = 1$. So we have

$$\Delta x = \frac{1-0}{n} = \frac{1}{n} \qquad \text{and} \qquad x_i = 0 + i\frac{1}{n} = \frac{i}{n}$$

$$\text{The exact area } = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$$

In general, we can find the exact area under the graph of $f(x)$ from $x = a$ to $x = b$ by

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\Delta x$$

$$\text{The exact area } = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x$$

Aside: Computing $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n}$ exactly.

$$\lim_{n\to\infty} \left[\left(\frac{1}{n}\right)^3 \frac{1}{n} + \left(\frac{2}{n}\right)^3 \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^3 \frac{1}{n} \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^4} \left(\underbrace{1^3 + 2^3 + 3^3 + \cdots + n^3}_{= \left(\frac{n(n+1)}{2}\right)^2} \right) \right]$$

factoring out $\frac{1}{n^4}$ in each term

formula in book

$$= \lim_{n\to\infty} \left[\frac{1}{n^4} \left(\frac{n(n+1)}{2} \right)^2 \right]$$

multiply this out and compute the limit. (It will be $1/4 = 0.25$)