7.1 - 7.5 Review - Integration Techniques

7.1: Integration By Parts

- Integration by Parts Formula: $\int u dv = uv - \int v du$ and $\int_a^b u dv = uv|_a^b - \int_a^b v du$
- Understand how to perform integration by parts and how to choose your u. Remember u should be 'easy' to differentiate and dv should be 'easy' to antidifferentiate. To help you decide we discussed LIPET (Logs, Inverse trig, Powers of x, Exponentials, Trig); let u the part of the integrand that occurs earlier in LIPET. Once you know the method, you shouldn't need LIPET anymore.
- There is no one rule that tells you when to use which method. But here are some instances when integration by parts tends to work:
  - Products - specifically powers of x times $\sin(x)$, $\cos(x)$, or e^x. It also works with e^x times $\sin(x)$ or $\cos(x)$ (but you have to pay attention to when the process repeats).
  - Logs - integrals involving logs (take $u = \ln(x)$)
  - Inverse Trig - integrals involving inverse trigonometric functions (take $u =$ 'inv. trig')

7.2:Trigonometric Integrals

- There is one general theme: Rewrite the integral using identities and use u-substitution (if needed). A review of trigonometry and these identities can be found in Appendix D of your textbook.
- The identities that are most useful are:

  $$\cos^2(x) + \sin^2(x) = 1 \text{ , which we may use as } \cos^2(x) = 1 - \sin^2(x) \text{ or } \sin^2(x) = 1 - \cos^2(x)$$

  Dividing these by $\cos^2(x)$ gives more useful identities:

  $$1 + \tan^2(x) = \sec^2(x) \text{ , which we may use as } 1 = \sec^2(x) - \tan^2(x) \text{ or } \tan^2(x) = \sec^2(x) - 1$$

  The rest of the identities stem from the sum identities, specifically:

  $$\begin{aligned} \sin(A)\cos(B) &= \tfrac{1}{2}[\sin(A-B) + \sin(A+B)] \\ \sin(A)\sin(B) &= \tfrac{1}{2}[\cos(A-B) - \cos(A+B)] \\ \cos(A)\cos(B) &= \tfrac{1}{2}[\cos(A-B) + \cos(A+B)] \end{aligned}$$

  You use these in full generality in Part 5 of the week 5 assignments (orthogonal functions). But we most often use these identities in special cases (specifically, when $A = B = x$) which are the **half angle identities**:

  $$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) \text{ , } \cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x) \text{ , and } \sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

- The textbook and my other reviews online summarize the cases. But for me, it is most useful to attempt to set up a u-substitution, this gives three possibilities for $\sin(x)$ and $\cos(x)$ and three possibilities for $\sec(x)$ and $\tan(x)$:

- Integrals with $\sin(x)$ and $\cos(x)$
  1. Using $u = \sin(x)$ and $du = \cos(x)dx$, look for an odd power of $\cos(x)$ and pull out one factor. Then use the identities to turn the rest of the problem into $\sin(x)$'s.
  2. Using $u = \cos(x)$ and $du = -\sin(x)dx$, look for an odd power of $\sin(x)$ and pull out one factor. Then use the identities to turn the rest of the problem into $\cos(x)$'s.
  3. If both powers are even, the above two methods don't work. Use the half angle identities to simplify the problem.
- Integrals with $\sec(x)$ and $\tan(x)$
  1. Using $u = \tan(x)$ and $du = \sec^2(x)dx$, look for an even power of $\sec(x)$ and pull out two factors. Then use the identities to turn the rest of the problem into $\tan(x)$'s.
  2. Using $u = \sec(x)$ and $du = \sec(x)\tan(x)dx$, look for an odd power of $\sec(x)\tan(x)$ and pull out one factor of $\sec(x)\tan(x)$. Then use the identities to turn the rest of the problem into $\sec(x)$'s.
  3. If neither of the above cases work, you may need to use other identities or integration by parts (read examples 7 and 8 in 7.2). Also it may be useful to have the following integrals (you are allowed to quote these in homework and exams):

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \text{ and } \int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C$$

7.3: Trigometric Substitution

- This is a different type of substitution. We let $x = a\sin(\theta)$, $x = a\tan(\theta)$, or $x = a\sec(\theta)$. These methods often work well for the specific integrals that have $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ somewhere in the problem.
- The basic idea: Replace x with a trigonometric function, do the substitution, use a trig identity and hope for the best. (the result is a trig. integral which we studied in 7.2). These steps will ensure that you get rid of the square root.
- The book and my other reviews reminding you of which trig. function to use along with the corresponding identity.
- At the end of the problem you will have a solution in terms of θ. To get back to x, you need to know the 'triangle trick' as is discussed in the text (and we will discuss it in class).
- For these type of substitutions there is a restriction for the variable θ which is necessary in order to ensure that we can solve for θ (that is so that the substitutions are invertible).

7.4: Partial Fractions

- The basic idea: Factor the denominator, then break up the fraction into separate fractions involving the factors of the original denominator. This is successfully because each individual part can be integrated easily.

  If we have a quadratic that factors, then we immediately factor it from the beginning. For example,

$$\frac{1}{x^2 - 6x + 5} = \frac{1}{(x-1)(x-5)}.$$

Then we will have a method for breaking this up and integration each part separately by using the first set of integrals below.

If we have a quadratic that doesn't factor, then we will need to complete the square and use what we know about the second integrals below. For example,

$$\frac{1}{x^2+x+1} = \frac{1}{x^2+x+1/4+1/4+1} = \frac{1}{(x+1/2)^2+5/4}.$$

Then we will use u-substitution along with the second and third integrals below.

- Thus, we are trying to simplify into integrals into an "easier form". We will use all the following facts (I used a substitution to get the last one):

$$\int \frac{1}{x-b}\,dx = \ln|x-b| + C \quad , \quad \int \frac{1}{(x-b)^2}\,dx = -\frac{1}{x-b} + C \quad , \quad \int \frac{1}{(x-b)^3}\,dx = -\frac{1}{2}\frac{1}{(x-b)^2} + C$$

$$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C, \quad \int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln|x^2+a^2| + C$$

- Here's how to do it:

  1. If the highest power on top is bigger than, or equal to, the highest power on the bottom: **DIVIDE**
  2. Factor the denominator where possible.
  3. Complete the square for quadratics that don't factor.
  4. Write out the general partial fraction expansion. For example:

$$\frac{BLAH}{(x+1)(x-3)^2((x-2)^2+7)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D(x-2)+E}{(x-2)^2+7}$$

  5. Multiply both sides by the common denominator and solve for A, B, C, D and E (either by expanding and comparing coefficients, or plugging in special values for x)
  6. Integrate each part separately using the integrals above.

Here are a couple extra examples:

- Simple linear factors:

$$\begin{aligned}\int \frac{x^2+2}{x^3+3x^2+2x}\,dx &= \int \tfrac{x^2+2}{x(x^2+3x+2)}\,dx = \tfrac{x^2+2}{x(x+1)(x+2)}\,dx \\ &= \int \tfrac{A}{x} + \tfrac{B}{x+1} + \tfrac{C}{x+2}\,dx\end{aligned}$$

Now we multiply the denominator out to get $x^2+2 = A(x+1)(x+2)+Bx(x+2)+Cx(x+1)$. We plug in strategic values for x to find A, B, and C.

$x=0$ gives $2=2A$, so $A=1$; $x=-1$ gives $3=-B$; $x=-2$ gives $6=2C$, so $C=3$.

Now we finish integrating

$$\int \frac{1}{x} + \frac{-3}{x+1} + \frac{3}{x+2}\,dx = \ln(x) - 3\ln(x+1) + 3\ln(x+2) + C.$$

– A quadratic factor:

$$\int \frac{x+1}{x^3+x^2+x}\,dx \begin{aligned}[t] &= \int \tfrac{x+1}{x(x^2+x+1)}\,dx = \int \tfrac{x+1}{x((x+1/2)^2+5/4)}\,dx \\ &= \int \tfrac{A}{x} + \tfrac{B(x+1/2)+C}{(x+1/2)^2+5/4}\,dx \end{aligned}$$

Now we multiply the denominator out to get $x+1 = A((x+1/2)^2+5/4)+(B(x+1/2)+C)x$. We plug in strategic values for x to find A, B, and C.

$x = 0$ gives $1 = (6/4)A$, so $A = 2/3$.

$x = -1/2$ gives $1/2 = A(5/4) + C(-1/2)$, so $1/2 = (2/3)(5/4) + C(-1/2)$, so $C = 2/3$.

Now there is not an obvious choice of x, but we can choose some other value (or equate coefficients) to find B.

$x = 1/2$ gives $3/2 = 2/3(9/4) + (B + 2/3)(1/2)$, so $0 = 1/2B + 1/3$, so $B = -2/3$.

Now we finish integrating: LET $u = x + 1/2$

$$\begin{aligned} \int \tfrac{2/3}{x} + \tfrac{-2/3(x+1/2)+2/3}{(x+1/2)^2+5/4}\,dx &= \tfrac{2}{3}\ln(x) + \int \tfrac{-2/3(x+1/2)+2/3}{(x+1/2)^2+5/4}\,dx. \\ &= \tfrac{2}{3}\ln(x) + \int \tfrac{-2/3u+2/3}{u^2+5/4}\,du \\ &= \tfrac{2}{3}\ln(x) - 2/3\int \tfrac{u}{u^2+5/4}\,du + 2/3\int \tfrac{1}{u^2+5/4}\,du \\ &= \tfrac{2}{3}\ln(x) - 2/3\ln(u^2+5/4) + 2/3\tfrac{1}{\sqrt{5/4}}\tan^{-1}\left(\tfrac{u}{\sqrt{5/4}}\right) + C \\ &= \tfrac{2}{3}\ln(x) - 2/3\ln((x+1/2)^2+5/4) + 2/3\tfrac{1}{\sqrt{5/4}}\tan^{-1}\left(\tfrac{x+1/2}{\sqrt{5/4}}\right) + C \end{aligned}$$

Choosing and Combining Methods

- The best way to learn which methods to use is to read 7.5 and PRACTICE LOTS OF PROBLEMS. Problems are readily available in your homework, textbook, and on the weekly assignment.