1. (12 points) Compute:

(a) $\displaystyle\int_0^{\pi^2} \sin\left(\frac{\sqrt{x}}{4}\right)\, dx.$

$t = \sqrt{x} \Leftrightarrow t^2 = x$

$2t\,dt = dx$

$$\int_0^{\pi} \sin\left(\frac{t}{4}\right) 2t\,dt$$

$u = 2t \qquad dv = \sin\left(\tfrac{1}{4}t\right)dt$

$du = 2dt \qquad v = -4\cos\left(\tfrac{1}{4}t\right)$

$$= -8t\cos\left(\frac{t}{4}\right)\Big|_0^{\pi} - \int_0^{\pi} -8\cos\left(\frac{t}{4}\right)dt$$

$$= \left[-8\pi\cos(\pi/4) + 8(0)\right] + 32\sin\left(\frac{t}{4}\right)\Big|_0^{\pi}$$

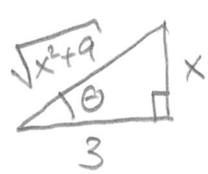
$$= -8\pi\frac{\sqrt{2}}{2} + 32\underbrace{\sin\left(\frac{\pi}{4}\right)}_{\sqrt{2}/2} - 0$$

$$\boxed{\begin{aligned} &= -4\pi\sqrt{2} + 16\sqrt{2} \\ &= 4\sqrt{2}(-\pi + 4) = 4\sqrt{2}(4 - \pi)\end{aligned}}$$

(b) $\displaystyle\int \frac{5}{(x^2+9)^{3/2}}\, dx.$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$



$$= \int \frac{5}{(9\sec^2\theta)^{3/2}} 3\sec^2\theta\, d\theta$$

$$= \int \frac{5}{\underset{9}{\cancel{27}}\sec^{\cancel{3}}\theta} \cancel{3\sec^2}\theta\, d\theta$$

$$= \frac{5}{9}\int \cos\theta\, d\theta$$

$$= \frac{5}{9}\sin\theta + C$$

$$= \boxed{\frac{5}{9}\frac{x}{\sqrt{x^2+9}} + C}$$

2. (12 points) Compute:

(a) $\displaystyle\int \frac{x-8}{x^3+4x^2}\,dx.$

$$\frac{x-8}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$$= \int \frac{3/4}{x} - \frac{2}{x^2} - \frac{3/4}{x+4}\,dx$$

$$x-8 = Ax(x+4)+B(x+4)+Cx^2$$

$$x=0 \Rightarrow -8 = B(4) \Rightarrow B=-2$$

$$= \boxed{\frac{3}{4}\ln|x| + \frac{2}{x} - \frac{3}{4}\ln|x+4| + C}$$

$$x-8 = Ax^2+4Ax+Bx+4B+Cx^2$$

$$A+C=0 \Rightarrow C=-A$$

$$4A+B=1 \Rightarrow 4A=1-B \Rightarrow 4A=3 \Rightarrow A=3/4$$

$$4B=-8 \qquad C=-3/4$$

(b) $\displaystyle\int \frac{1}{\sqrt{x^2+6x+5}}\,dx$

$$x^2+6x+9-9+5$$

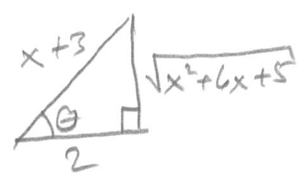
$$=(x+3)^2-4$$

$$\int \frac{1}{\sqrt{(x+3)^2-4}}\,dx$$

$$x+3 = 2\sec\theta$$

$$dx = 2\sec\theta\tan\theta\,d\theta$$

$$\int \frac{1}{2\tan\theta}\,2\sec\theta\tan\theta\,d\theta$$



$$=\int \sec\theta\,d\theta$$

$$=\ln|\sec\theta+\tan\theta|+C$$

$$\boxed{=\ln\left|\frac{x+3}{2}+\frac{\sqrt{x^2+6x+5}}{2}\right|+C} = \ln|x+3+\sqrt{x^2+6x+5}|+D$$

$$D = C-\ln(2)$$

3. (12 points) Compute:

(a) $\int \ln(1+x^2)\,dx.$

$u = \ln(1+x^2) \qquad dv = dx$

$du = \frac{2x}{1+x^2}\,dx \qquad v = x$

$$= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

$$\begin{array}{r} 2 \\ x^2+1 \overline{\big)\, 2x^2} \\ -(2x^2+2) \\ \hline -2 \end{array}$$

$$= x\ln(1+x^2) - \int 2 - \frac{2}{1+x^2}\,dx$$

$$= \boxed{x\ln(1+x^2) - 2x + 2\tan^{-1}(x) + C}$$

(b) $\int_0^{\pi/8} \sec^4(2x)\tan^3(2x)\,dx.$

$\boxed{\text{OPTION 1}}$ $u = \tan(2x) \quad du = 2\sec^2(2x)\,dx$

$$\int_0^{\pi/8} (\tan^2(2x)+1)\tan^3(2x)\sec^2(2x)\,dx$$

$$= \int_0^1 (u^2+1)\,u^3\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_0^1 u^5 + u^3\,du$$

$$= \frac{1}{12}u^6 + \frac{1}{8}u^4 \Big|_0^1$$

$$= \frac{1}{12} + \frac{1}{8} = \frac{2+3}{24} = \boxed{\frac{5}{24}}$$

$\boxed{\text{OPTION 2}}$ $u = \sec(2x) \quad du = 2\sec(2x)\tan(2x)\,dx$

$$\int_0^{\pi/8} \sec^3(2x)\,(\sec^2(2x)-1)\sec(2x)\tan(2x)\,dx$$

$$= \int_1^{\sqrt{2}} u^3(u^2-1)\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_1^{\sqrt{2}} u^5 - u^3\,du$$

$$= \frac{1}{12}u^6 - \frac{1}{8}u^4 \Big|_1^{\sqrt{2}}$$

$$= \left(\frac{1}{12}\cdot 8 - \frac{1}{8}\,4\right) - \left(\frac{1}{12} - \frac{1}{8}\right)$$

$$= \frac{7}{12} - \frac{3}{8} = \frac{14-9}{24} = \boxed{\frac{5}{24}}$$

4. (12 points)

(a) Use Simpson's rule with $n = 4$ to approximate the average value of $f(x) = \frac{e^x}{x}$ on the interval from $x = 1$ to $x = 9$.
(You do not need to simplify your answer, put all the numbers in the correct places and leave it expanded out)

$\Delta x = \frac{9-1}{4} = 2 \qquad x_0 = 1, x_1 = 3, x_2 = 5, x_3 = 7, x_4 = 9$

$$\text{AVERAGE VALUE} = \frac{1}{9-1} \int_1^9 \frac{e^x}{x}\, dx$$

$$\boxed{\frac{1}{8} \cdot \frac{1}{3} \cdot \underset{\Delta x}{2} \left[\frac{e^1}{1} + 4\frac{e^3}{3} + 2\frac{e^5}{5} + 4\frac{e^7}{7} + \frac{e^9}{9} \right]}$$

≈ 134.65

ACTUAL ≈ 129.498

(b) Find the arc length of the curve $y = \frac{1}{3}x^{3/2}$ from $x = 0$ to $x = 12$.

$$y' = \frac{1}{3}\frac{3}{2}x^{1/2} = \frac{1}{2}\sqrt{x}$$

$$\text{LENGTH} = \int_0^{12} \sqrt{1 + \left(\tfrac{1}{2}\sqrt{x}\right)^2}\, dx$$

$$= \int_0^{12} \sqrt{1 + \tfrac{1}{4}x}\, dx$$

$u = 1 + \frac{1}{4}x$
$du = \frac{1}{4}dx$
$dx = 4\,du$

$$= \int_1^4 \sqrt{u}\; 4u$$

$$= 4 \cdot \tfrac{2}{3} u^{3/2} \Big|_1^4$$

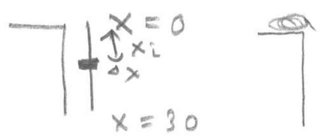
$$= \tfrac{8}{3}\left(4^{3/2} - 1^{3/2}\right) = \tfrac{8}{3}(8-1) = \boxed{\tfrac{56}{3}} \approx 18.\overline{6}$$

5. (12 points) **For the problems below include units in your final answers.**

(a) A 30 meter cable with density $\frac{1}{4.9}$ kg/m is ~~having~~ hanging over the side of a tall building. How much total work is done in lifting the cable ~~half-way up~~?
(Remember, the acceleration due to gravity is $g = 9.8$ m/s^2).

OPTION 1

LABEL TOP $x=0$



$\text{FORCE} = 9.8 \cdot \frac{1}{4.9}\,\Delta x = 2\Delta x$

$\text{DIST} = x_i$

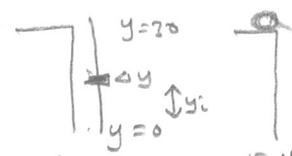
$\int_0^{30} 2x\,dx$

$= x^2 \Big|_0^{30}$

$= 30^2 = \boxed{900 \text{ Joules}}$

OPTION 2

LABEL BOTTOM $y=0$



$\text{FORCE} = 9.8\,\frac{1}{4.9}\,\Delta y = 2\Delta y$

$\text{DIST} = 30 - y_i$

$\int_0^{30} 2(30-y)\,dy$

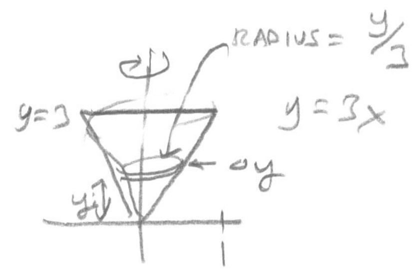
$= 60y - y^2 \Big|_0^{30}$

$= 60(30) - (30)^2 = 1800 - 900$

$= \boxed{900 \text{ Joules}}$

(b) The portion of the graph $y = 3x$ between $x = 0$ feet to $x = 1$ feet is rotated around the y-axis to form a container (so the container is a cone). The container is full of a liquid that has density 90 lbs/ft^3.
Find the work required to pump all of the liquid out over the side of the container.

FORCE (HORIZONTAL SLICE) $= 90 \cdot \pi \left(\frac{y_i}{3}\right)^2 \Delta y$



DIST. $= 3 - y_i$

$\int_0^3 90\pi \,\frac{y^2}{9}\,(3-y)\,dy$

$10\pi \int_0^3 3y^2 - y^3\,dy$

$10\pi \left(y^3 - \frac{1}{4}y^4 \Big|_0^3\right)$

$10\pi \left[27 - \frac{81}{4}\right] = 10\pi\left[\frac{108-81}{4}\right] = 10\pi \cdot \frac{27}{4} =$

$= \boxed{\frac{270\pi}{4} = \frac{135\pi}{2}}$ ft - lbs