Chapter 9: Basic Differential Equation Applications

A *differential equation* is an equation involving derivatives. These equations often involve both the independent variable, say t, the dependent variable, say y (which is a function of t), and the derivative $\frac{dy}{dt}$, and possibly higher derivatives. The full study of differential equations and how to work with them takes several quarters including the classes Math 307, Math 309, Math 435, and Math 436. Our goal in Math 125 is to introduce you to basic examples and discuss how to solve separable equations which only involves methods of integration.
The goal of this overview sheet is to summarize and expand on examples of differential equations (in addition to in class and textbook examples). I hope this helps you gain comfort with differential equations and how they are used in modeling some basic scenarios. You certainly are expected to be able to set up the differential equations in these basic types of examples, but mostly I created this sheet for your own interest.

Example 1: The Law of Natural Growth
A function $y(t)$ satisfies *the law of natural growth* if "The rate of change of y is proportional to the value of y", which translate directly to the differential equation

$$\frac{dy}{dt} = ky \text{ , for some relative growth constant } k.$$

We call k the *relative growth rate*. It is NOT the rate of change of $y(t)$. The rate of change of the $y(t)$ is $\frac{dy}{dt}$, which is not a constant. But it is a measure of the *relative* rate which can be seen by rewriting the equation as $k = \frac{dy/dt}{y}$ which is the ratio of the rate of change with respect to the value of y.

Applications:

1. Population Growth: This is a common model for unrestricted population growth. Usually we use the notation, $P(t)$, for the size of population at time t and $P(0) = P_0$ is the initial population and k is the *relative* population growth rate.

2. Radioactive Decay: A radioactive isotope loses mass over time. The function, $y(t)$, for the mass at time t satisfies the *the law of natural decay*, which is the same as the law of natural growth (just the k will be negative). Often information is given in terms of *half-life*, which gives an initial condition. For example, if half-life is 100 years, then you know that $y(100) = \frac{1}{2}y(0)$ which we use as an initial condition to solve for k.

3. Bank Accounts: If an initial amount $A(0) = P_0$ is deposited into some account with decimal annual rate r, compounded n times a year, then the amount in t years is given by $A(t) = P_0(1 + \frac{r}{n})^{nt}$ (We don't discuss this formula, I just give it to you for background). In the limit as $n \to \infty$ we get what we call *continuous compounding* and $\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r$, so the formula becomes $A(t) = P_0e^{rt}$.

   By differentiating, we find the function for the amount $A(t)$ in an account with annual decimal rate r, compounded continuously satisfies the law of natural growth so that $\frac{dA}{dt} = rA$.

   Note: This is only the model for the situation where you invest a certain initial amount and never deposit more or withdraw more from the account. (To account for this we would have to write down how the rate is effected which would change the differential equation).

Example 2: Shifted Natural Growth or Decay
This is not formal terminology, but a lot of basic problems involve a differential equation satisfying "The rate of change of y is proportional to the value of y along with some other constant growth/decay rate" which translate directly to the differential equation

$$\frac{dy}{dt} = ky + c, \text{ for some constants } k \text{ and } c.$$

Applications:

1. Population Growth with immigration: This is another common model for unrestricted population growth. But it also assumes that each year the population increases or decreases at an additional constant rate of c (meaning c people per year are entering the country or leaving (or dying) and that this rate is constant). An example: "assume the population of a city satisfies the law of natural growth and, in addition, approximately 1000 people per year move into the city". The differential equation would be $\frac{dy}{dt} = ky + 1000$.

2. Newton's Law of Cooling: It states "The rate of change of temperature an object is proportional to the difference between the object and the surrounding temperature." If $T(t)$ is the temperature at time t for some object and T_s is the fixed temperature of the surroundings, then Newton's law of cooling states that

$$\frac{dT}{dt} = k(T - T_s), \text{ for some cooling constant } k.$$

3. Adding/Removing Money from Bank Accounts: Assuming an account grows with a certain annual rate r, compounded continuous and additional money is deposited or withdrawn at a continuous constant rate of c dollars per year, then we get a differential equation of the form above.

4. Mixing Problems: Here is a fairly standard and full example. A 100 liter vat of Brine (salt-water) contains a certain amount of salt diluted in it. Let $y(t)$ be the amount of salt at time t. If a brine mixture that is 3 kg of salt per liter is pumped in at 8 liters per minute and the mixed vat is emptied at the same rate, then the general differential equation satisfied by $y(t)$ will have the form
$\frac{dy}{dt}$ = RATE IN - RATE OUT, where
RATE IN = 8 L/min 3 kg/L = 24 kg/min salt coming in
RATE OUT = 8 L/min $\frac{y(t)}{100}$ kg/L = 8y/100 kg/min salt going out
Thus,

$$\frac{dy}{dt} = 24 - \frac{8y}{100}$$

5. Basic Circuit Examples: A law from circuits (Kirchhoff's voltage law) states the sum of voltage drops is zero around a circuit. You will need to see your book for examples (because you need a circuit diagram). The current, $i(t)$, in the circuit at time t can be determined by using information about voltage across resistors (which is proportional to $i(t)$), across inductors (which is proportional to $i'(t)$), across capacitors (proportional to an integral of $i(t)$) and the voltage source $V(t)$. If there is no capacity and voltage is constant, we often get differential equations of the form $Ri + Li' = V$, so $L\frac{di}{dt} = -Ri + V$ (and if L, R and V are constants then the differential equation is of the form above). In these scenarios we would have to give the differential equation, I am not expecting you to be able to derive it (this is not a circuits class).

Example 3: The Logistic Equation (Restricted Growth):
In these types of models, the function grows much like natural growth until it approaches some carrying capacity that it cannot exceed.
Applications:

1. Population Growth: A common model for a population that is restricted to a maximum size (carrying capacity) of M is the so-called logistic equation which is:
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \text{ , for some constant } k$$
Notice that when $P = M$ we get $\frac{dP}{dt} = 0$ (so the population is unchanging (flat) if $P(t) = M$). If the population is smaller then M, then the slope is positive, so the population increases. But as the population size approaches M, the slope gets closer to zero, so the population approaches the size M asymptotically. This discussion shows you how you can predict information from the differential equation alone. Now see if you can solve the differential equation by separating and doing some partial fractions.

2. Spread of a Rumor: As you saw in your worksheet, if $y(t)$ is the number of people that know a rumor at time t and there are M people total, then the spread of the rumor can be modeled by $\frac{dy}{dt} = ky(M - y)$ ("the rate of the spread of the rumor is proportional the product of those that have heard the rumor and those that haven't heard it"). This is very similar to the logistic equation (if you factor out M is actually the same form).

3. Spread of a Disease: This can also be modeled by the logistic equation with M being the total number of people (there is this limit to the number of people that can get the disease). There are other models with different features that can be used as well. As an example consider the Gompertz model which uses the differential equation $\frac{dy}{dt} = ky\ln\left(\frac{M}{y}\right)$. Note that as y approaches M the slope $\frac{dy}{dt}$ approaches 0 and we have similar general behavior to the logistic equation. (This model approaches the asymptotes in a different manner, more gradual, and makes it a better model for some situations.)

Example 4: Gravity and Air Resistance:
Here is a quick discussion of models that account for air resistance.
Applications:

1. Only Gravity (no air resistance): We've done this many times, if $s(t)$ is position, $v(t) = s'(t)$ is velocity, and $a(t) = v'(t) = s''(t)$ is acceleration. With gravity $g = 9.8$ we get the differential equation, $s''(t) = v'(t) = -9.8$, which we can integrate twice to get $s(t)$.

2. Gravity and air resistance: Various models are used depending on the object that is traveling through the air. A fairly accurate model for lower density falling objects is the model $v' = -g + kv$, so the air resistance is proportional to the velocity. For higher densities and speeds, the model $v' = -g + kv^2$ is often used, so the air resistance is proportional to the square of the velocity. For a golf ball, your textbook mentions that experiments have show the model $v' = -g + kv^{1.3}$ seems to work well. So the differential equation to model motion taking into account for air resistance involves a term proportional to some function of the velocity of the object (and this will depend on the size, density and shape of the moving object).

   You can use these models to find *terminal velocity* for a falling object along with answer other projectile questions including air resistance.