5.5 Substitution

The substitution rule says, if \( u = g(x) \), then

\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

We often remember this, by writing \( du = g'(x)dx \). Here let me discuss some common questions.

1. This is really just the chain rule from differential calculus. Remember that chain rule says that

\[
\frac{d}{du}(F(u)) = f(u), \text{ then } \frac{d}{dx}(F(g(x))) = f(g(x))g'(x).
\]

In terms of integrals, this says

\[
\text{If } F(u) + C = \int f(u) \, du, \text{ then } F(g(x)) + C = \int f(g(x))g'(x) \, dx.
\]

So we really are just undoing the chain rule and the notation \( u = g(x) \) and \( du = g'(x)dx \) helps us see this in an organized way.

2. In terms of the definition of the derivative remember that

\[
\int_a^b f(g(x))g'(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x.
\]

If we want to change the variable to \( u = g(x) \). Then we are in fact ‘transforming’ the function, and interval, into a new coordinates system. Instead of the \( xy \)-coordinate system, it will be the \( uy \)-coordinate system. So the question becomes how does that effect the rectangles and areas we are computing.

Graph \( u = g(x) \) and look at one of your subdivisions, then label \( u_{i-1} = g(x_{i-1}) \) and \( u_i = g(x_i) \).

The slope between these two points on the graph of \( u = g(x) \) would be \( \frac{u_i - u_{i-1}}{x_{i} - x_{i-1}} = \frac{\Delta u}{\Delta x} \). If the interval is small, then this slope would be very close to the slope of the tangent line \( g'(x_i) \). Thus, we have \( \Delta u \approx g'(x_i) \), which we can write as \( \Delta u \approx g'(x_i)\Delta x \), with increasing accuracy as \( \Delta x \) gets smaller. (I am just giving a plausibility argument, this is not mathematically rigorous.)

In any case, going back to the definition of the integral we have

\[
\int_a^b f(g(x))g'(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f(u_i)\Delta u = \int_{g(a)}^{g(b)} f(u) \, du.
\]

This is what we are thinking about when we write \( u = g(x) \) and \( du = g'(x)dx \).

3. In terms of practicalities, when faced with an integral that isn’t in our list of integrals we already know, your job is to pick \( u = g(x) \) and compute \( du = g'(x)dx \) and HOPE! You hope that making the change gives an integral that is in our list of integrals. Here are common things to try:

(a) Look for \( u = \text{‘inside function’}, \text{ and } du = \text{‘outside function’} \, dx \).
(b) Look for \( u = \ln(x) \), with \( \frac{1}{x} \) appearing elsewhere in the problem.
(c) Look for \( u = \text{‘denominator’} \) with the derivative of \( u \) in the numerator.
(d) Even if things don’t line up perfectly, try \( u = \text{‘inside function’}, \) and if not all the \( x \)'s cancel, then go back and solve for \( x \) in your substitution and see if that helps.