Chapter 5 Review

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections. Please inform me if you find any typos on this sheet.

1. Approximating Net Areas with Rectangles (Riemann Sums)
   - Understand the way to develop more and more accurate net area approximations by using the following development. Here we try to find the area between the $x$-axis and the graph of $f(x)$ from $x = a$ to $x = b$.
     (a) Subdivide the interval into $n$ sub-intervals of equal width. $\text{WIDTH} = \frac{b-a}{n}$
     (b) Choose a point $x^*_i$ in each sub-interval and plug it into the function to get the height of each rectangle. $\text{HEIGHT} = f(x^*_i)$
     (c) The approximate net area is the sum of the areas of each rectangle ($\text{WIDTH} \times \text{HEIGHT}$).

   Approximate Net Area $= f(x^*_1)\Delta x + f(x^*_2)\Delta x + \cdots + f(x^*_n)\Delta x = \sum_{i=1}^{n} f(x^*_i)\Delta x$

   (d) The exact net area is the value we get in the limit as we put in an infinite number of rectangles:

   Exact Net Area $= \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i)\Delta x$

2. The Definite Integral (Riemann Integral)
   - Having a formula for the exact net area, we defined a simpler notation called the definite integral of $f(x)$ from $x = a$ to $x = b$:

   $\int_{a}^{b} f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i)\Delta x$

   - The integral represents the “net area”. That is:

   $\int_{a}^{b} f(x)\,dx = \left( \text{area of stuff above the x-axis} \right) - \left( \text{area of stuff below the x-axis} \right)$

   - We did examples of all of these properties in class:
     $\int_{a}^{b} cdx = (b - a)c$ (Integral of a constant)
     $\int_{a}^{b} (f(x) + g(x))\,dx = \int_{a}^{b} f(x)\,dx + \int_{a}^{b} g(x)\,dx$ (Breaking up a sum)
     $\int_{a}^{b} cf(x)\,dx = c \int_{a}^{b} f(x)\,dx$ (Pulling out a constant)
     $\int_{b}^{a} f(x)\,dx = -\int_{a}^{b} f(x)\,dx$ (Flipping endpoints)
     $\int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx = \int_{a}^{c} f(x)\,dx$ (Areas of Consecutive Regions)

3. The Fundamental Theorem of Calculus
   - Part 1: If $f$ is continuous on $[a, b]$, then

   $g(x) = \int_{a}^{x} f(t)\,dt \Rightarrow g'(x) = f(x)$
Part 2: If $f$ is continuous on $[a, b]$ and $F$ is an antiderivative of $f$, then
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- Understand how to use Part 1. You often have to use either (1) the chain rule or (2) breaking up the interval.
- We will use Part 2 a lot this quarter. To use it, you (1) Find an antiderivative, (2) Plug the endpoints into the antiderivative, and (3) Subtract.

4. The Indefinite Integral
- The Fundamental Theorem gives the precise way that antiderivatives, derivatives, and areas are related. Thus, if we wish to study general antidifferentiation it makes sense to use the integral notation. So we define the indefinite integral of $f(x)$ by:
$$\int f(x)\,dx = \text{the general antiderivative of } f(x)$$

Don’t forget to use the ‘+C’ when working with general antiderivatives.
- The integral of a rate of change is the net change. Understand what this interpretation of the fundamental theorem means and how it is used in practice.
- Understand the difference between net change (displacement) and total change.
  - \[ \text{Displacement} = \text{net change} = \int_a^b f(t)\,dt. \]
  - \[ \text{Total Change} = \int_a^b \left| f(t)\right|\,dt \]
    In order to compute this integral you must (1) find where $f(t) = 0$, (2) Break up the integral into separate integrals for each time $f(t)$ changes from above/below the $x$-axis, and (3) Evaluate each integral and take the areas as a positive numbers.

5. The Substitution Rule (\( u \)-substitution)
- Practice, Practice, Practice!!! It is very important to learn this techniques. Ultimately, integration is somewhat more of an art form than other mathematical techniques you have learned. It is vital that you learn all the rules of integration well and are willing to be flexible, patient and experimental when trying to evaluate an integral.
- The Substitution Rule, or $u$-substitution technique, is essential the chain rule in reverse and is stated as follows for the indefinite and definite integrals, respectively:
  - If $u = g(x)$ and $du = g'(x)\,dx$, then
    $$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$
  - If $u = g(x)$ and $du = g'(x)\,dx$, then
    $$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$
- In practice, you use the $u$-substitution as follows:
  (a) Choose the function $u = g(x)$. (This is the art form part, you have to make the choice. If your choice doesn’t work, you will have to start over. Be patient and practice)
  (b) Compute $du = g'(x)\,dx$, $g(a)$, and $g(b)$
  (c) Replace $g(x)$ with $u$, $dx$ with $\frac{du}{g'(x)}$, and replace the endpoints. Also simplify if you can.
  (d) If all your $x$’s vanish from the problem, you have successfully done a $u$-substitution. See if you can integrate what’s left. If not all $x$’s vanish, then you need to start over and choose a different $u$. 

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• You need to start looking at an integral as a whole. Look for the following:
  
  (a) *A function and its derivative somewhere in the integral.* For example: if you see $x^4$ in one part of the function, look for $x^3$ somewhere else. If you do see $x^3$, then perhaps u-substitution will work with $u = x^4$.
  
  (b) *A function inside another function.* In these situations, it is often a good idea to take $u = $ the inside function.

6. Miscellaneous

• Understand how antiderivatives can be used to solve problems involving acceleration, velocity and distance.