1. (12 pts) Evaluate

(a) $\int \frac{\ln(x)}{x^3}\,dx$

BY PARTS

$u = \ln(x) \qquad dv = x^{-3}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{2}x^{-2}$

$= -\frac{1}{2}x^{-2}\ln(x) - \int -\frac{1}{2}x^{-3}\,dx$

$= -\frac{\ln(x)}{2x^2} + \int \frac{1}{2}x^{-3}\,dx$

$= -\frac{\ln(x)}{2x^2} + -\frac{1}{4}x^{-2} + C$

$= \boxed{-\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} + C}$

ALSO CORRECT: $-\frac{1}{2}x^{-2}\ln(x) - \frac{1}{4}x^{-2} + C$ OR $-\frac{(2\ln(x)+1)}{4x^2} + C$

(b) $\int \frac{\tan^2(x)\sec^2(x)}{\cos^2(x)}\,dx$

TRIG

EVEN SEC

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$= \int \tan^2(x)\sec^4(x)\,dx$

$= \int \tan^2(x)\underbrace{\sec^2(x)}_{(\tan^2(x)+1)}\sec^2(x)\,dx$

$= \int u^2(u^2+1)\,du$

$= \int u^4 + u^2\,du = \frac{1}{5}u^5 + \frac{1}{3}u^3 + C$

$= \boxed{\frac{1}{5}\tan^5(x) + \frac{1}{3}\tan^3(x) + C}$

2. (12 pts) Evaluate

(a) $\int \frac{x^2+16}{x^3+2x^2}\,dx$

PARTIAL FRAC

$$\frac{x^2+16}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$\Rightarrow x^2+16 = Ax(x+2) + B(x+2) + Cx^2$$

$$x=0 \Rightarrow 16 = B(2) \Rightarrow B = 8$$

$$x=-2 \Rightarrow 20 = C(4) \Rightarrow C = 5$$

COEF. OF x^2: $1 = A + C$

SO $A = 1 - C = 1 - 5 = -4$

$$\int \frac{-4}{x} + \frac{8}{x^2} + \frac{5}{x+2}\,dx$$

$$= \boxed{-4\ln|x| - \frac{8}{x} + 5\ln|x+2| + C}$$

(b) $\int \frac{4\cancel{x}}{(x^2+2x+5)^{3/2}}\,dx$

TRIG SUB

$x^2+2x+1-1+5$

$(x+1)^2+4$

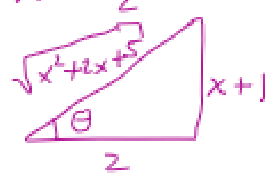
$x+1 = 2\tan\theta \Rightarrow 4\tan^2\theta + 4 = 4\sec^2\theta$

$dx = 2\sec^2\theta\, d\theta$

$$= 4\int \frac{2\tan\theta - 1}{(4\sec^2\theta)^{3/2}}\, 2\sec^2\theta\, d\theta$$

$$= \frac{\cancel{4}\,2}{\cancel{8}} \int \frac{2\tan\theta - 1}{\sec^{\cancel{3}}\theta}\, \cancel{\sec^2\theta}\, d\theta$$

$\tan(\theta) = \frac{x+1}{2}$



$$= \int \frac{2\tan\theta}{\sec\theta} - \frac{1}{\sec\theta}\, d\theta$$

$$= \int 2\sin\theta - \cos\theta\, d\theta$$

$$= -2\cos\theta - \sin\theta + C$$

$$= \boxed{-2\cdot\frac{2}{\sqrt{x^2+2x+5}} - \frac{(x+1)}{\sqrt{x^2+2x+5}} + C} = \frac{-4-x-1}{\sqrt{x^2+2x+5}} + C$$

$$\frac{-(x+5)}{\sqrt{x^2+2x+5}} + C$$

3. (12 pts) Evaluate

(a) $\displaystyle\int_1^5 \frac{\sqrt{x-1}}{x+3}\,dx$

SUBSTITUTION JUST LIKE HW

$t = \sqrt{x-1}$

$t^2 = x-1$

$t^2+1 = x$

$2t\,dt = dx$

$$= \int_0^2 \frac{t}{(t^2+1)+3}\,2t\,dt$$

$$= \int_0^2 \frac{2t^2}{t^2+4}\,dt$$

$$t^2+4 \overline{)2t^2} = 2, \quad -(2t^2+8) \to -8$$

$$= \int_0^2 2 - \frac{8}{t^2+4}\,dt$$

$$= 2t - 8\cdot\tfrac{1}{2}\tan^{-1}\left(\tfrac{t}{2}\right)\Big|_0^2$$

$$= \left(4 - 4\tan^{-1}(1)\right) - \left(0 - \underbrace{4\tan^{-1}(0)}_{0}\right)$$

$$= 4 - 4\cdot\pi/4$$

$$= \boxed{4-\pi}$$

(b) $\displaystyle\int \sqrt{9-x^2}\,dx$

TRIG SUB

$x = 3\sin\theta \rightarrow \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta$

$dx = 3\cos\theta\,d\theta$

$$= \int 3\cos\theta\cdot 3\cos\theta\,d\theta$$

$$= 9\int \cos^2\theta\,d\theta$$

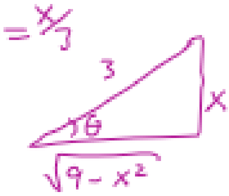
$$= \frac{9}{2}\int 1+\cos(2\theta)\,d\theta$$

$$= \frac{9}{2}\left[\theta + \tfrac{1}{2}\sin(2\theta)\right] + C$$

RECALL:

$\frac{1}{2}\sin(2\theta) = \cos\theta\sin\theta$

$\sin\theta = x/3$



$$= \frac{9}{2}\left[\theta + \cos\theta\sin\theta\right] + C$$

$$= \boxed{\frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) + \frac{\sqrt{9-x^2}}{3}\cdot\frac{x}{3}\right] + C}$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9-x^2} + C$$

4. (12 pts) The two parts below are unrelated.

(a) Evaluate the improper integral $\int_1^\infty \frac{1+\sqrt{x}}{x^2}\,dx$. $\longrightarrow$ $\frac{1}{x^2} + \frac{x^{1/2}}{x^2}$

$$\lim_{t\to\infty} \int_1^t x^{-2} + x^{-3/2}\,dx$$

$$= \lim_{t\to\infty} \left[-x^{-1} + -2x^{-\frac{1}{2}} \Big|_1^t \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{x} - \frac{2}{\sqrt{x}} \Big|_1^t \right]$$

$$= \lim_{t\to\infty} \left[\underbrace{\left(-\frac{1}{t} - \frac{2}{\sqrt{t}}\right)}_{\to 0} - (-1-2) \right]$$

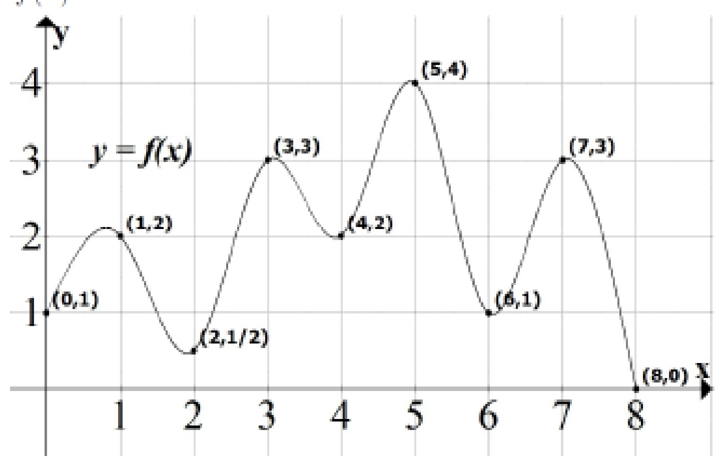
$$= \boxed{3}$$

(b) Consider the region under the function $y = f(x)$ from $x = 0$ to $x = 8$ which is shown below:

Correctly use the values from the graph to answer the following:
Use Simpson's rule with $n = 4$ subdivisions to estimate the volume of the solid obtained by rotating this region about the x-axis. In other words, estimate

$$\int_0^8 \pi(f(x))^2\,dx.$$

Note: Your final answer will be number times π, you can leave it in that form.



$$\Delta x = \frac{8-0}{4} = 2,\ x_0 = 0,\ x_1 = 2,\ x_2 = 4,\ x_4 = 6,\ x_5 = 8$$

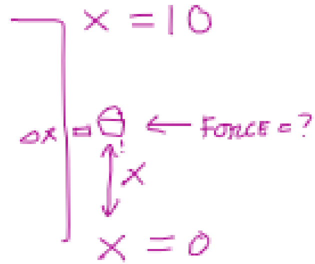
$$\frac{1}{3}\Delta x \left[\pi(f(0))^2 + 4\pi(f(2))^2 + 2\pi(f(4))^2 + 4\pi(f(6))^2 + 2\pi(f(8))^2 \right]$$

$$\frac{1}{3}(2)\left[\pi(1)^2 + 4\pi\left(\frac{1}{2}\right)^2 + 2\pi(2)^2 + 4\pi(1)^2 + 2\pi(0)^2 \right]$$

$$\frac{2}{3}\pi\left[1 + 1 + 8 + 4\right] = \frac{2}{3}\pi \cdot 14 = \boxed{\frac{28\pi}{3}}$$

5. (12 pts) A leaky 2 lbs bucket is lifted from the ground to a height of 10 ft at a constant speed with a rope that weighs 0.3 lbs/ft. Initially the bucket contains 20 pounds of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 10-ft level.

(a) How much work is done in lifting the leaky bucket alone (meaning not including the work to lift the rope)?



$x = 0 \Rightarrow$ FORCE $= 2 + 20 = 22$ lbs

$x = 10 \Rightarrow$ FORCE $= 2$ lbs

FORCE $= m(x - x_1) + y_1$ $\quad m = \frac{22-2}{0-10} = -2$

$= -2(x-0) + 22$

$F = -2x + 22$ lbs $\quad D = \Delta x$

$$W = \int_0^{10} -2x + 22\, dx$$

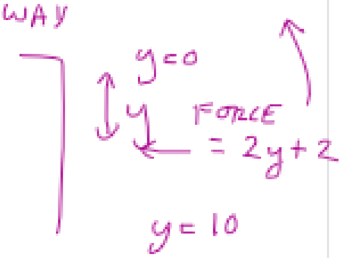
$$= -x^2 + 22x \Big|_0^{10}$$

$$= \left(-(10)^2 + 22(10)\right) - (0) = -100 + 220$$

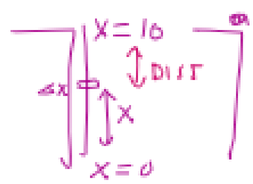
$$= \boxed{120 \text{ ft-lbs}}$$

NOTE
COULD ALSO LABEL THIS WAY

$\int_0^{10} 2y + 2\,dy = 120$



(b) How much work is done in lifting the rope (meaning not including the work to lift the bucket)?



FORCE $= 0.3\Delta x$

DIST $= 10 - x$

$$W = \int_0^{10} (10 - x)0.3\,dx$$

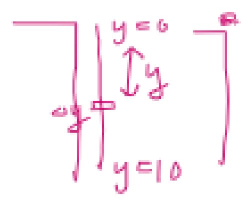
$$= \int_0^{10} 3 - 0.3x\,dx$$

$$= 3x - \frac{0.3}{2}x^2 \Big|_0^{10}$$

$$= (30 - 0.15 \cdot (100)) - 0$$

$$= 30 - 15 = \boxed{15 \text{ ft-lbs}}$$

ALSO COULD LABEL



$$\int_0^{10} y \cdot 0.3\,dy$$

$$= \frac{1}{2}y^2 0.3 \Big|_0^{10}$$

$$= 0.15 \cdot 100 = 15 \text{ ft-lbs}$$

(c) How much work is done all together to lift the bucket and rope.

Total Work = 135 foot-pounds