FROM BEGINNING OF 7.2 LECTURE

Challenge integrals with solutions

① $\int \cos(x)\ln(\cos(x))\,dx$

$u=\ln(\cos(x))$ $\quad dv=\cos(x)dx$

$du=\frac{-\sin(x)}{\cos(x)}dx$ $\quad v=\sin(x)$

$= \sin(x)\ln(\cos(x)) - \int -\frac{\sin^2(x)}{\cos(x)}dx$ (by parts)

$= \sin(x)\ln(\cos(x)) + \int \frac{1-\cos^2(x)}{\cos(x)}dx$ (identity)

$= \sin(x)\ln(\cos(x)) + \int \sec(x)-\cos(x)\,dx$

$= \boxed{\sin(x)\ln(\cos(x)) + \ln|\sec(x)+\tan(x)| - \sin(x) + C}$

② $\int x e^x \ln(x-1)\,dx$

BY PARTS

$u=\ln(x-1)$ $\quad dv=xe^x dx$

$du=\frac{1}{x-1}dx$ $\quad v=(x-1)e^x$

$=(x-1)e^x\ln(x-1) - \int \frac{1}{x-1}(x-1)e^x dx$

$=\boxed{(x-1)e^x\ln(x-1) - e^x + C}$

ASIDE $\int xe^x dx$ $\quad u=x$ $\quad dv=e^x dx$

$du=dx$ $\quad v=e^x$

$=xe^x-\int e^x dx$

$=xe^x-e^x+C$

$=(x-1)e^x+C$

③ $\int \sin(x)\cos(x)\,e^{\cos(x)}dx$

$t=\cos(x)$

$dt=-\sin(x)dx$

$dx=-\frac{1}{\sin(x)}dt$

$=\int \cancel{\sin(x)}\,t e^t \frac{1}{-\cancel{\sin(x)}}dt$

$=-\int te^t dt$

$= -(t-1)e^t+C$, $= \boxed{-(\cos(x)-1)e^{\cos(x)}+C}$

④ $\int x^4\tan^{-1}(x)\,dx$

BY PARTS

$u=\tan^{-1}(x)$ $\quad dv=x^4 dx$

$du=\frac{1}{1+x^2}dx$ $\quad v=\frac{1}{5}x^5$

$=\frac{1}{5}x^5\tan^{-1}(x)-\frac{1}{5}\int\frac{x^5}{1+x^2}dx$

$t=1+x^2$ $\quad x^2=t-1$

$dt=2x\,dx$

$dx=\frac{1}{2x}dt$

$=\frac{1}{5}x^5\tan^{-1}(x)-\frac{1}{5}\int\frac{\cancel{x^5}\,x^4}{t}\frac{1}{2\cancel{x}}dt$

$=\frac{1}{5}x^5\tan^{-1}(x)-\frac{1}{10}\int\frac{(t-1)^2}{t}dt$ $\quad x^4=(t-1)^2$

$=\frac{1}{5}x^5\tan^{-1}(x)-\frac{1}{10}\int\frac{t^2-2t+1}{t}dt$

$=\frac{1}{5}x^5\tan^{-1}(x)-\frac{1}{10}\int t-2+\frac{1}{t}dt$

$=\frac{1}{5}x^5\tan^{-1}(x)-\frac{1}{20}t^2+\frac{2}{10}t-\frac{1}{10}\ln|t|+C$

$=\boxed{\frac{1}{5}x^5\tan^{-1}(x)-\frac{1}{20}(1+x^2)^2+\frac{1}{5}(1+x^2)-\frac{1}{10}\ln|1+x^2|+C}$

NEED TO DO INTEGRATION BY PARTS 5 TIMES.

⑥ $\int x^5 \sin(x)\,dx$

$u = x^5 \qquad dv = \sin(x)dx$

$du = 5x^4 \qquad v = -\cos(x)$

$$= -x^5\cos(x) - \int (5x^4)(-\cos(x))dx$$

$u = 5x^4 \qquad dv = -\cos(x)dx$

$du = 20x^3 \qquad v = -\sin(x)$

$$= -x^5\cos(x) - \left[-5x^4\sin(x) - \int 20x^3(-\sin(x))dx\right]$$

$$= -x^5\cos(x) + 5x^4\sin(x) + \int (20x^3)(-\sin(x))dx$$

$u = 20x^3 \qquad dv = -\sin(x)dx$

$du = 60x^2dx \qquad v = \cos(x)$

$$= -x^5\cos(x) + 5x^4\sin(x) + 20x^3\cos(x) - \int 60x^2\cos(x)dx$$

$u = 60x^2 \qquad dv = \cos(x)dx$

$du = 120x\,dx \qquad v = \sin(x)$

$$= -x^5\cos(x) + x^4\sin(x) + 20x^3\cos(x) - \left[60x^2\sin(x) - \int 120x\sin(x)dx\right]$$

$$= -x^5\cos(x) + 5x^4\sin(x) + 20x^3\cos(x) - 60x^2\sin(x) + \int 120x\sin(x)dx$$

$u = 120x \qquad dv = \sin(x)dx$

$du = 120dx \qquad v = -\cos(x)$

$$= -x^5\cos(x) + 5x^4\sin(x) + 20x^3\cos(x) - 60x^2\sin(x) - 120x\cos(x) - \int 120(-\cos(x))dx$$

$$= \boxed{-x^5\cos(x) + 5x^4\sin(x) + 20x^3\cos(x) - 60x^2\sin(x) - 120x\cos(x) + 120\sin(x) + C}$$

You will NEVER HAVE TO DO THIS MANY STEPS ON AN EXAM.

OFTEN, people save a lot of writing by doing all the derivatives and integrations from the by parts at first in a table on the right. (You can see, from how I set it up, that each u is the derivative of the previous and each v is the integral of the previous). Then keep track of alternating sign. This is called tabular integration by parts (and you don't need to know it for this class).