Page 1(a)

1. (12 pts) Evaluate

(a) $\int \frac{5x^2+x+8}{x^3+4x}\,dx$

$$\frac{5x^2+x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$ 1

$$\Longrightarrow 5x^2+x+8 = A(x^2+4)+(Bx+C)x$$

$x=0 \Rightarrow 8 = 4A \Rightarrow A = 2$ 2

COEF. OF x^2: $5 = A + B \Rightarrow B = 3$

COEF. OF x: $1 = C$

$$\int \frac{2}{x} + \frac{3x+1}{x^2+4}\,dx = \int \frac{2}{x}\,dx + \int \frac{3x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$ 1

2

Answer = $2\ln|x| + \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

1 **-0.0**
Correct

2 **-6.0**
No Correct Work (no attempt at partial fractions)

3 **-5.0**
Significant errors in set up or form (such as wrong decomposition)

4 **-2.0**
Multiple constants wrong in solving for decomposition

5 **-1.0**
Error in one of the constants

6 **-2.0**
Integration Error

7 **-1.0**
Arithmetic Error

8 **-0.5**
Small miscopy error

9 **-0.5**
Missing $+C$

Page 1(b)

(b) $\int \frac{1}{\sqrt{4x^2+16x+52}}\,dx$

$4(x^2+4x+4-4+13)$ 1

$4((x+2)^2+9)$

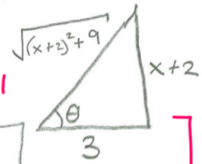
$= \int \frac{1}{2}\,\frac{1}{\sqrt{(x+2)^2+9}}\,dx$

$x+2 = 3\tan\theta \Rightarrow \sqrt{9(\tan^2\theta+1)} = 3\sec\theta$ 1

$dx = 3\sec^2\theta\,d\theta$

$= \frac{1}{2}\int \frac{1}{3\sec\theta}\,3\sec^2\theta\,d\theta$ 1

$= \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$ 1



$= \frac{1}{2}\ln\left|\frac{\sqrt{(x+2)^2+9}}{3} + \frac{x+2}{3}\right| + C$

$= \frac{1}{2}\ln\left|\sqrt{x^2+4x+13} + x+2\right| \underbrace{-\frac{1}{2}\ln(3) + C}_{D}$

Answer = $\frac{1}{2}\ln\left|\sqrt{x^2+4x+13} + x+2\right| + D$ 2

1 -0.0
Correct

2 -6.0
No attempt at completing the square or using trig sub.

3 -5.0
Correctly completed the square but major set-up errors after that.

4 -1.0
Errors in completing the square

5 -4.0
Errors in trig substitution.

6 -3.0
Correctly substituted and got to $\frac{1}{2}\int \sec(\theta), d\theta$ but errors after that.

7 -2.0
Correctly got to $\frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|$, but errors in using triangle and returning to x

8 -2.0
Did not use triangle, gave a final answer that involved $\sec(\tan^{-1}((x+2)/3))$, as stated in class and on the cover of the exam, points will be deducted for such an answer).

9 -1.0
Arithmetic or substitution error

0 -0.5
Small miscopy error.

-0.5
Forgetting $+C$

Page 2(a)

2. (12 pts) Evaluate

(a) $\int x^2 \tan^{-1}(x)\,dx$

$u = \tan^{-1}(x) \quad dv = x^2\,dx$] 2

$du = \frac{1}{x^2+1}\,dx \quad v = \frac{1}{3}x^3$] 1

$= \frac{1}{3}x^3\tan^{-1}(x) - \frac{1}{3}\int \frac{x^3}{x^2+1}\,dx$

$$\begin{array}{r} x \\ x^2+1 \overline{)\, x^3 } \\ -(x^3+x) \\ \hline -x \end{array}$$

$= \frac{1}{3}x^3\tan^{-1}(x) - \frac{1}{3}\int x - \frac{x}{x^2+1}\,dx$] 2

$= \frac{1}{3}x^3\tan^{-1}(x) - \frac{1}{6}x^2 + \frac{1}{3}\cdot\frac{1}{2}\ln(x^2+1) + C$] 1

Answer = $\frac{1}{3}x^3\tan^{-1}(x) - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + C$

1	-0.0	Correct
2	-6.0	No Correct Work
3	-5.0	Missing Main Concept (tried some method other than by parts and didn't make progress)
4	-4.0	Significant errors in set up (such as wrong choice for u)
5	-3.0	Correctly did by parts, but didn't make progress with the integral $\int \frac{x^3}{x^2+1}\,dx$.
6	-2.0	Error in long division or simplification of $\int \frac{x^3}{x^2+1}\,dx$.
7	-1.0	Arithmetic or integration error.
8	-0.5	Small miscopy error
9	-0.5	Forgot $+C$

Page 2(b)

(b) $\int \frac{x+4}{x(1+\sqrt{x})^2}\,dx$

$t=\sqrt{x} \Rightarrow t^2 = x$

$2t\,dt = dx$

2

$$\int \frac{(t^2+4)}{\cancel{t^2}(1+t)^2}\, 2\cancel{t}\,dt = \int \frac{2t^2+8}{t(t+1)^2}\,dt$$

$$\frac{2t^2+8}{t(t+1)^2} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{(t+1)^2}$$

1

$2t^2+8 = A(t+1)^2 + Bt(t+1) + Ct$

$t=0 \Rightarrow 8 = A$

$t=-1 \Rightarrow 10 = -C \Rightarrow C=-10$

COEF. OF t^2: $2 = A+B \Rightarrow B = 2-A = -6$

2

$$\int \frac{8}{t} + \frac{-6}{t+1} + \frac{-10}{(t+1)^2}\,dt = 8\ln|t| - 6\ln|t+1| + \frac{10}{t+1} + C$$

1

Answer = $\underbrace{8\ln(\sqrt{x})}_{=4\ln(x)} - 6\ln(\sqrt{x}+1) + \frac{10}{\sqrt{x}+1} + C$

1	-0.0	All Correct
2	-6.0	No Correct Work
3	-5.0	Missing Main Concept (tried something, but didn't correctly use substitution to turn it into a partial fraction problem)
4	-4.0	Substituted and got to $\int \frac{2t^2+8}{t(t+1)^2}\,dt$, but made major errors in partial fraction set up.
5	-2.0	Errors in finding constants for partial fractions.
6	-1.0	Error in one one constant for partial fractions.
7	-1.0	Error in integration at the end.
8	-1.0	Forgot to give final answer in terms of x
9	-0.5	Small miscopy error
0	-0.5	Forgot $+C$

Page 3(a)

12

3. ([redacted] pts) Evaluate

(a) $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

$x = 3\sec\theta \rightarrow \sqrt{9(\sec^2\theta - 1)} = 3\tan\theta$ 1

$dx = 3\sec\theta\tan\theta\,d\theta$

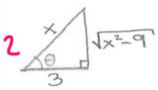
$= \int \frac{3\tan\theta}{27\sec^3\theta}\, 3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int \underbrace{\frac{\tan^2\theta}{\sec^2\theta}}_{\sin^2\theta}\,d\theta$ 2

$= \frac{1}{3}\int \sin^2\theta\,d\theta = \frac{1}{6}\int 1 - \cos(2\theta)\,d\theta$ 1

$= \frac{1}{6}\theta - \frac{1}{12}\sin(2\theta) + C$ $\qquad \sin(2\theta) = 2\cos\theta\sin\theta$

$= \frac{1}{6}\theta - \frac{1}{6}\cos\theta\sin\theta + C$

$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\,\frac{3}{x}\,\frac{\sqrt{x^2-9}}{x}$ 2



Answer = $\frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$

1	-0.0	Correct
2	-6.0	No Correct Work (no attempt at trig substitution)
3	-5.0	Knew that $x = 3\sec(\theta)$, but multiple errors in use and set up after that.
4	-4.0	Correctly simplified to $\int \frac{\tan^2(\theta)}{\sec^2(\theta)}\,d\theta$, but was unable to simplify and made no progress after that.
5	-3.0	Correctly got to $\int \sin^2(\theta)\,d\theta$ and used the half-angle identity, but errors after that.
6	-2.0	Errors in using triangle or didn't use half-angle identity correctly (also you lose these points if you gave a final answer that involved $\sin(2\sec^{-1}(x/3))$ as stated in class and on the cover of the exam, points will be deducted for such an answer).
7	-1.0	Error in last step of answer (such as misinterpreting the triangle or error in writing final answer)
8	-0.5	Small miscopy error
9	-0.5	Forgot $+C$
0	+1.0	Drew triangle for trig, giving one point back for this despite other errors.

Page 3(b)

(b) Find the average value of $f(x) = x^2 e^{x/2}$ from $x = 0$ to $x = 4$.

1

$$\frac{1}{4-0}\int_0^4 x^2 e^{\frac{1}{2}x}\,dx$$

$u = x^2 \quad dv = e^{\frac{1}{2}x}dx$

$du = 2x\,dx \quad v = 2e^{\frac{1}{2}x}$

2

$$\frac{1}{4}\left[2x^2 e^{\frac{1}{2}x}\Big|_0^4 - \int_0^4 4x e^{\frac{1}{2}x}dx\right]$$

$u = 4x \quad dv = e^{\frac{1}{2}x}dx$

$du = 4dx \quad v = 2e^{\frac{1}{2}x}$

$$\frac{1}{4}\left[(32e^2 - 0) - \left(8x e^{\frac{1}{2}x}\Big|_0^4 - \int_0^4 8e^{\frac{1}{2}x}dx\right)\right]$$

2

$$\frac{1}{4}\left[\underbrace{32e^2 - (32e^2 - 0)}_{} + \int_0^4 8e^{\frac{1}{2}x}dx\right]$$

$$\frac{1}{4}\left[0 + 16e^{\frac{1}{2}x}\Big|_0^4\right] = \frac{1}{4}\left[16e^2 - 16\right]$$

1

Average Value = $4e^2 - 4$

1	-0.0	All Correct
2	-6.0	No Correct Work
3	-1.0	Forgot to compute average value (i.e. didn't divide integral by 4).
4	-4.0	Major errors in using by parts (such as wrong choice of u or set up).
5	-3.0	Got to $\frac{1}{4}\left(32e^2 - \int_0^4 4xe^{x/2}, dx\right)$, but didn't make progress in doing by parts again.
6	-2.0	Errors in derivatives or integrals in doing by parts (such as multiple wrong coefficients or evaluations)
7	-1.0	Integration or algebra error, such as sign error in integration by parts formula or simplification error.
8	-0.5	Small miscopy error

Page 4(a)

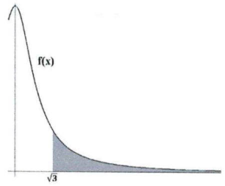
i. Evaluate the following integral to find the area: $\int_{\sqrt{3}}^{\infty} \frac{1}{x^2+1}\,dx$.

$\lim_{t\to\infty}\left(\int_{\sqrt{3}}^{t} \frac{1}{x^2+1}\,dx\right)$] 2

$\lim_{t\to\infty}\left(\tan^{-1}(x)\Big|_{\sqrt{3}}^{t}\right)$] 1

$\lim_{t\to\infty}\left(\tan^{-1}(t) - \tan^{-1}(\sqrt{3})\right)$] 1

$\underbrace{\frac{\pi}{2}}_{1} - \underbrace{\pi/3}_{1} = \pi/6$



Answer (or say diverges) = $\pi/6$

1	-0.0 All Correct
2	-6.0 No Correct Work
3	-2.0 Did NOT write as a limit
4	-4.0 Wrote as a limit, but made no progress in integration.
5	-2.0 Got to $\lim_{t\to\infty}\left(\tan^{-1}(t) - \tan^{-1}(\sqrt{3})\right)$, but didn't give the correct value of this limit.
6	-1.0 Couldn't figure out the value of $\tan^{-1}(\sqrt{3}) = \pi/3$. So did not give final answer in form $\pi/2 - \pi/3 = \pi/6$.
7	-0.5 Small miscopy error

Page 4(b)

ii. Set up an improper integral that represents the volume of the solid obtained by rotating this region around the y-axis (using shells!). Then evaluate the integral.

$$\lim_{t\to\infty} \int_{\sqrt{3}}^{t} 2\pi \frac{x}{x^2+1}\,dx \quad]\ 1$$

$$= 2\pi \lim_{t\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_{\sqrt{3}}^{t} \quad]\ 1$$

$$= \pi \lim_{t\to\infty}\left(\ln(t^2+1)-\ln(4)\right) = \infty \quad]\ 1$$

Integral Representing the Volume = $\int_{\sqrt{3}}^{\infty} 2\pi x \cdot \frac{1}{x^2+1}\,dx$] 2

Evaluation of integral (or say diverges) = DIVERGES] 1

1 -0.0
All Correct

2 -6.0
No Correct Work

3 -2.0
Errors in setting up volume integral, should be $\int_{\sqrt{3}}^{\infty} 2\pi \frac{x}{x^2+1}\,dx$.

4 -4.0
Correctly set up the volume integral, but no correct work in evaluating that integral.

5 -3.0
Wrote integral as a limit, but errors in integrating.

6 -2.0
Correctly got to $\lim_{t\to\infty} \pi\left(\ln(t^2+1)-\ln(4)\right)$, but made the wrong conclusion. This diverges!

7 -1.0
Arithmetic or integration error

8 -0.5
Small miscopy error

Page 5(a)

(a) Given that $\int_1^9 g(x)\,dx = \cancel{10}\ 6$, find $\int_1^3 xg(x^2) + 3x^2\,dx$

$= \underbrace{\int_1^3 x g(x^2)dx}_{+2} + \underbrace{\int_1^3 3x^2 dx}_{+2}$

$t = x^2$

$dt = 2x dx$

$= \frac{1}{2}\int_1^9 g(u)du \quad + \quad x^3\Big|_1^3 \quad = \frac{1}{2}\cdot 6 + 27 - 1$

Answer = 29

1 +0.0

Correct

2 +4.0

Correct

3 +2.0

Knew to separate and computed

$\int_1^3 3x^2\,dx = x^3\Big|_1^3 = 26$

4 +1.0

Small error in second term integration, so only 1 out of 2 for that part.

5 +2.0

Knew to use substitution ($u = x^2$) on first term and got

$\int_1^3 xg(x^2)\,dx = \int_1^9 \frac{1}{2}g(u)\,du = \frac{1}{2}\cdot 6 = 3$

6 +1.0

Did substitution on first term, but made error in your work so you only get 1 out of 2 for this part.

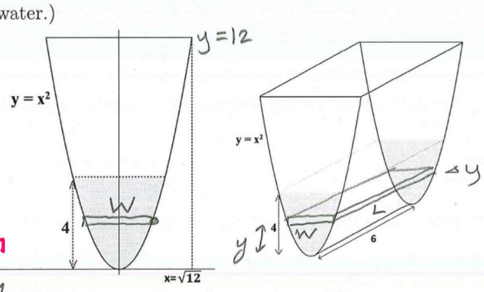
Page 5(b)

(b) A trough-shaped tank of length 6 feet has vertical cross sections given by $y = x^2$ (as shown). The top of the tank is at $x = \pm\sqrt{12}$ ft. The tank initially has water up to a height of 4 feet. Find the work required to pump the water to the top of the tank. (Use 62.5 lbs/ft^3 for the density of water.)

FOR A SLICE AT y OF THICKNESS Δy, WE HAVE

DIST LIFTED $= 12 - y$

FORCE $= 62.5 \cdot 6 \cdot W \cdot \Delta y$



WORK

$$= \underbrace{\int_0^4}_{3} \underbrace{(12-y)\,62.5 \cdot 6 \cdot 2\sqrt{y}}_{3}\,dy$$

$$= 750 \int_0^4 (12-y)\sqrt{y}\,dy$$

$$= 750 \int_0^4 12y^{1/2} - y^{3/2}\,dy$$

$$= 750 \left(8y^{3/2} - \tfrac{2}{5}y^{5/2}\Big|_0^4\right)$$

$$= 750\left(64 - \tfrac{2}{5} \cdot 32\right)$$

$$= 750 \cdot 64 \cdot \left(1 - \tfrac{1}{5}\right) = 750 \cdot 64 \cdot \tfrac{4}{5}$$

$$= 38{,}400 \text{ ft-lbs}$$

2

$W = 2\sqrt{y}$

SINCE

Work (include units) = 38,400 ft-lbs

1	-0.0	Correct
2	-8.0	No Correct Work
3	-1.0	Missing 62.5 or 6 from pattern for force, should look like $62.5 \cdot 6 \cdot W \cdot dy$, and you need to find W
4	-2.0	Error in width of cross-sections pattern should be $2\sqrt{y}$
5	-2.0	Error in distance lifted pattern should be $(12 - y)$
6	-1.0	Error in bounds should be $\int_0^4$
7	-2.0	Error in integration.
8	-1.0	Error in evaluation
9	-4.0	Errors in form, such as not showing understanding that you are finding a pattern in terms of \$y\$ for force times distance.
0	-0.5	Error in units (should be ft-lbs)