1. (12 pts) Evaluate

(a) $\int x \sec^4(x^2)\, dx$

$t = x^2$

$dt = 2x\, dx$

$\frac{1}{2} dt = x\, dx$

$= \frac{1}{2} \int \sec^4(t)\, dt$

$= \frac{1}{2} \int (\tan^2(t) + 1) \sec^2(t)\, dt$

$u = \tan(t)$

$du = \sec^2(t)\, dt$

$= \frac{1}{2} \int u^2 + 1\, du$

$= \frac{1}{2}\left(\frac{1}{3} u^3 + u\right) + C$

$= \boxed{\frac{1}{6} \tan^3(x^2) + \frac{1}{2} \tan(x^2) + C}$

(b) $\int \frac{x^2 - 2x + 12}{x^3 + 3x^2}\, dx$

$\frac{x^2 - 2x + 12}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$

$\Rightarrow x^2 - 2x + 12 = Ax(x+3) + B(x+3) + Cx^2$

$x = 0 \Rightarrow \quad 12 = 3B \Rightarrow B = 4$

$x = -3 \Rightarrow 9 + 6 + 12 = 9C \Rightarrow C = 3$

COEF. OF x^2: $1 = A + C \Rightarrow A = 1 - C$

$\Rightarrow A = -2$

$= \int \frac{-2}{x} + \frac{4}{x^2} + \frac{3}{x+3}\, dx$

$= \boxed{-2\ln|x| - \frac{4}{x} + 3\ln|x+3| + C}$

2. (12 pts) Evaluate

(a) $\int \tan^{-1}(x)\,dx$

$u = \tan^{-1}(x) \qquad dv = dx$

$du = \frac{1}{x^2+1}\,dx \qquad v = x$

$= x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{1}{u}\,du$

$= x\tan^{-1}(x) - \frac{1}{2}\ln|u| + C$

$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C}$

$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) + C}$

(b) $\int \frac{x^2}{(9-x^2)^{3/2}}\,dx$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$(9-x^2)^{3/2} = (9-9\sin^2\theta)^{3/2}$

$= (9(1-\sin^2\theta))^{3/2}$

$= (9\cos^2\theta)^{3/2}$

$= 27\cos^3\theta$

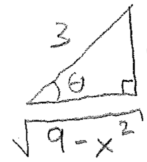
$= \int \frac{9\sin^2\theta}{27\cos^3\theta}\,3\cos\theta\,d\theta$

$= \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

$\sin\theta = \frac{x}{3}$



$= \boxed{\frac{x}{\sqrt{9-x^2}} - \sin^{-1}\left(\frac{x}{3}\right) + C}$

3. (12 pts) Evaluate

(a) $\displaystyle\int \frac{1}{\sqrt{x^2+6x+5}}\,dx$

$x^2+6x+9-9+5$

$(x+3)^2-4$

$x+3 = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$\sqrt{4\sec^2\theta-4} = \sqrt{4\tan^2\theta}$

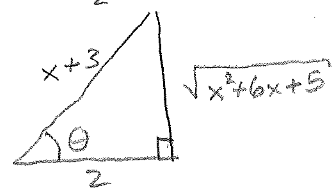
$= 2\tan\theta$

$$= \int \frac{1}{\cancel{2\tan\theta}}\, \cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$= \ln|\sec\theta+\tan\theta| + C$$

$$= \boxed{\ln\left|\frac{x+3}{2} + \frac{\sqrt{x^2+6x+5}}{2}\right| + C}$$

$\sec\theta = \frac{x+3}{2}$



(b) $\displaystyle\int_1^8 \frac{x^{1/3}}{x+x^{1/3}}\,dx.$

$t = x^{1/3}$

$t^3 = x$

$3t^2\,dt = dx$

$$= \int_1^2 \frac{t}{t^3+t}\cdot 3t^2\,dt$$

$$= \int_1^2 \frac{3t^3}{t^3+t}\,dt = \int_1^2 \frac{3t^{\cancel{3}\,2}}{\cancel{t}(t^2+1)}\,dt$$

$$= \int_1^2 \frac{3t^2}{t^2+1}\,dt$$

$$\begin{array}{r|l} & 3 \\ \hline t^2+1 & 3t^2 \\ & -(3t^2+3) \\ \hline & -3 \end{array}$$

$$= \int_1^2 3 - \frac{3}{t^2+1}\,dt$$

$$= 3t - 3\tan^{-1}(t)\Big|_1^2 = (6-3\tan^{-1}(2)) - (3-3\tan^{-1}(1))$$

$$= \boxed{3 - 3\tan^{-1}(2) + \frac{3\pi}{4}}$$

4. (12 pts) The two parts below are unrelated.

(a) Find the **average value** of the function $f(x) = (\ln(x))^2$ on the interval $x = 1$ to $x = e$. (Your final answer will involved the number e, leave it in simplified exact form).

$$\frac{1}{e-1}\int_1^e (\ln(x))^2\,dx \qquad u=(\ln(x))^2 \quad dv=dx$$

$$du=\frac{2\ln(x)}{x}dx \quad v=x$$

$$\frac{1}{e-1}\left[x(\ln(x))^2\Big|_1^e-\int_1^e 2\ln(x)dx\right]$$

$$u=2\ln(x) \quad dv=dx$$

$$\frac{1}{e-1}\left[(e-0)-\left(2x\ln(x)\Big|_1^e-\int_1^e 2\,dx\right)\right] \qquad du=\frac{2}{x}dx \quad v=x$$

$$\frac{1}{e-1}\left[e-(2e-0)+2x\Big|_1^e\right]$$

$$\frac{1}{e-1}\left[-e+(2e-2)\right]=\frac{e-2}{e-1}$$

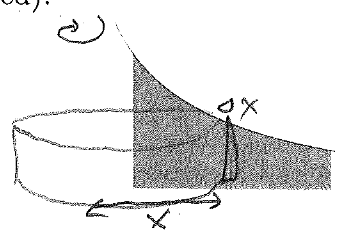
Average value of $f(x)$ = $\boxed{\frac{e-2}{e-1}}$ ≈ 0.418

(b) The region shown is bounded by the curves $y = e^{-\sqrt{x}}$, $y = 0$, $x = 1$ and $x = 5$. Assume this region is rotated about the y-axis to form a solid.
Set up the integral for the volume of the solid (do not evaluate).

Then use Simpson's rule with $n = 4$ to approximate the value of the integral. (Leave the Simpson's rule approximation expanded and unsimplified).

$$\Delta x=\frac{5-1}{4}=1 \qquad x_0=1,\ x_1=2,\ x_2=3$$

$$x_3=4,\ x_5=5$$



Volume Integral = $\int_1^5 2\pi x\, e^{-\sqrt{x}}\,dx$

Approx =

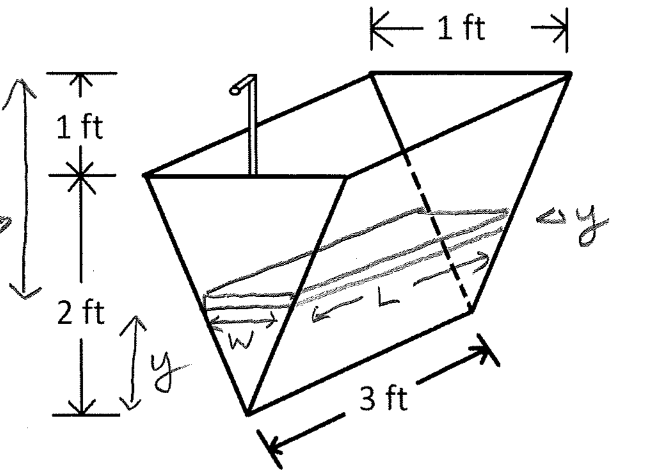
$$\boxed{2\pi\cdot\frac{1}{3}\cdot 1\cdot\left[e^{-1}+4\cdot 2e^{-\sqrt{2}}+2\cdot 3e^{-\sqrt{3}}+4\cdot 4e^{-2}+5e^{-\sqrt{5}}\right]}$$

≈ 12.7216

5. (12 points) The trough-shaped tank shown is full of water and all the water is going to be pumped up and out of a spout. The dimensions are as indicated in the picture. Note the top of the spout is 1 foot above the top of the full tank.

Find the work required to pump all the water out of the spout (Use 62.5 lbs/ft^3 for the density of water.)

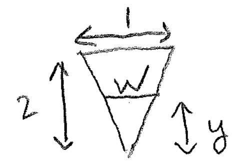
LET $y=0$ BE THE BOTTOM.
CONSIDER A HORIZONTAL SLICE AT HEIGHT y AND THICKNESS Δy.
FOR ANY y WITH $0 \leq y \leq 2$, WE FIND THE PATTERN FOR...



DIST. LIFTED $= 3-y$

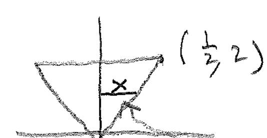
$$\text{FORCE} = 62.5 \cdot L \cdot W \cdot \Delta y$$
$$= 62.5 \cdot 3 \cdot W \cdot \Delta y$$
$$= 62.5 \cdot 3 \cdot \tfrac{1}{2} y \, \Delta y$$

TO FIND THE WIDTH PATTERN



$$\frac{W}{y} = \frac{1}{2} \Rightarrow \boxed{W = \tfrac{1}{2}y}$$

OR USE



slope $= \frac{2}{1/2} = 4$

LINE: $y = 4x$
$\Rightarrow x = \frac{1}{4}y$
AND $W = 2x$
$= \boxed{\tfrac{1}{2}y}$

$$\text{WORK} = \int_0^2 (3-y) \cdot 62.5 \cdot \tfrac{3}{2} y \, dy$$
$$= 62.5 \cdot \tfrac{3}{2} \int_0^2 3y - y^2 \, dy$$
$$= 62.5 \cdot \tfrac{3}{2} \left[\tfrac{3}{2}y^2 - \tfrac{1}{3}y^3 \Big|_0^2 \right]$$
$$= 62.5 \cdot \tfrac{3}{2} \left[6 - 8/3 \right]$$
$$= 62.5 \cdot [9-4]$$
$$= 62.5 \cdot 5$$
$$= 312.5$$

Work (include units) = $\boxed{312.5 \text{ ft-lbs}}$