9.1 Intro to Differential Equations

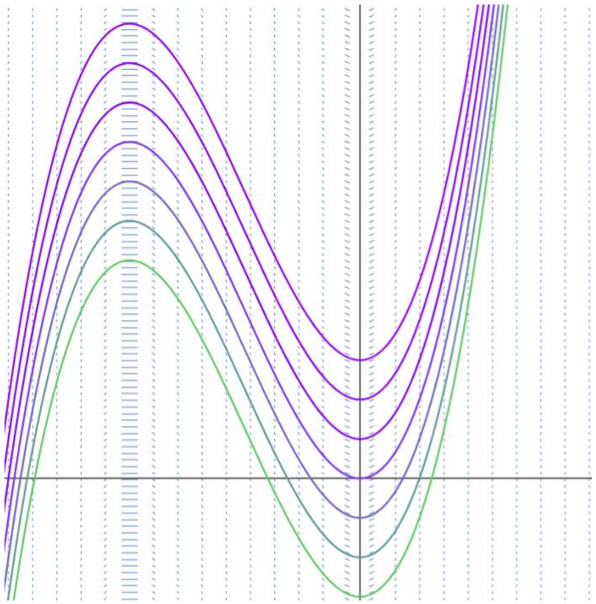
A **differential equation** is an equation involving derivatives.

A **solution to a differential equation** is any function that satisfies the equation.

Entry Task:
Find $y = y(x)$ such that

$$\frac{dy}{dx} - 8x = x^2 \text{ and } y(0) = 5.$$

Check your final answer



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***What is a solution?* and**
How to check solutions?

Example 1

Consider the differential equation:

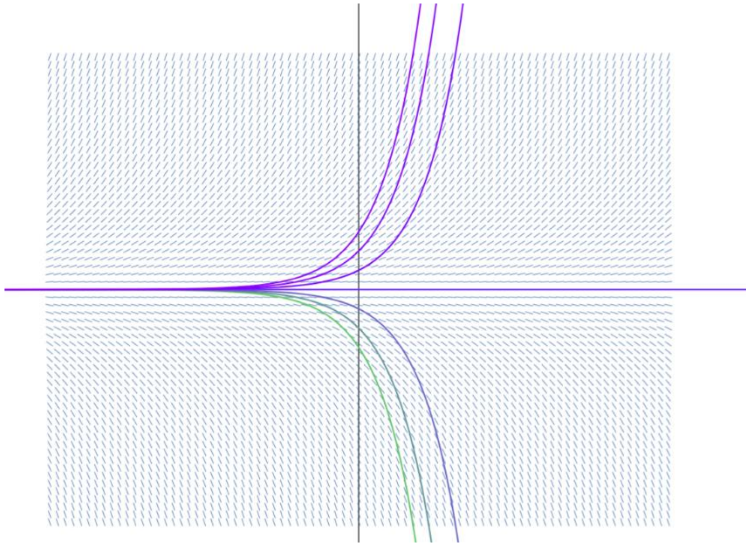
$$\frac{dP}{dt} = 2P$$

(a) Is P(t) = $8e^{2t}$ a solution?

(b) Is P(t) = t^3 a solution?

(c) Is P(t) = 0 a solution?

The **general solution** to $\frac{dP}{dt} = 2P$ is $P(t) = Ce^{2t}$, for any constant C.



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Example 2

Consider the 2nd order differential equation

$$y'' + 2y' + y = 0.$$

(a) Is $y = e^{-2t}$ a solution?

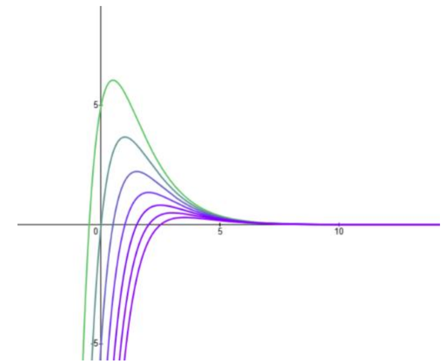
(b) Is $y = t\,e^{-t}$ a solution?

(c) There is a sol'n that looks like $y = e^{rt}$. Can you find *r*?

Side Note *(NOT in m125, but is in m207)*

A 2nd order differential equation tends to have two unknown constants, the general solution in this example looks like

$$y = (C_1 + C_2 t)e^{-t}$$



Q: Can you think of a 2nd order equation we have solved in this class?

9.1 HW example….

3. [– / 7 Points]

(a) What can you say about a solution of the equation $y' = -\frac{1}{3}y^2$ just by looking at the differential equation?

- ◯ The function y must be increasing (or equal to 0) on any interval on which it is defined.
- ◯ The function y must be decreasing (or equal to 0) on any interval on which it is defined.
- ◯ The function y must be strictly increasing on any interval on which it is defined.
- ◯ The function y must be equal to 0 on any interval on which it is defined.
- ◯ The function y must be strictly decreasing on any interval on which it is defined.

(b) Verify that all members of the family $y = \frac{3}{(x + C)}$ are solutions of the equation in part (a).

We substitute the values of y and y' and test the solution to see if the left hand side (LHS) is equal to the right hand side (RHS).

$$y = \frac{3}{x + C} \Rightarrow y' = -\frac{\square}{(x + C)^2}$$

$$\text{LHS} = y' = -\frac{\square}{(x + C)^2} = -\frac{1}{3}\left(\frac{\square}{x + C}\right)^2 = -\frac{1}{3}y^2 = \text{RHS}$$

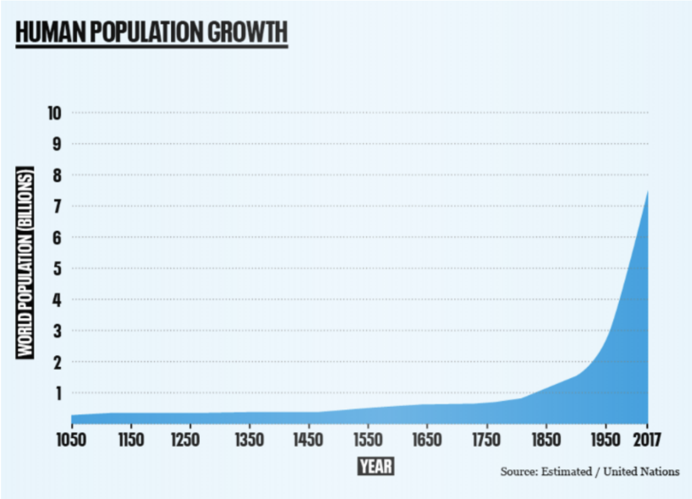
(c) Can you think of a solution of the differential equation $y' = -\frac{1}{3}y^2$ that is not a member of the family in part (b)?

- ◯ $y = x$ is a solution of $y' = -\frac{1}{3}y^2$ that is not a member of the family in part (b).
- ◯ $y = e^{3x}$ is a solution of $y' = -\frac{1}{3}y^2$ that is not a member of the family in part (b).
- ◯ $y = 3$ is a solution of $y' = -\frac{1}{3}y^2$ that is not a member of the family in part (b).
- ◯ Every solution of $y' = -\frac{1}{3}y^2$ is a member of the family in part (b).
- ◯ $y = 0$ is a solution of $y' = -\frac{1}{3}y^2$ that is not a member of the family in part (b).

(d) Find a solution of the initial-value problem.

$$y' = -\frac{1}{3}y^2 \qquad y(0) = 0.1$$

Motivation and Fun Examples



1. Natural Unrestricted population

P(t) = population at year *t*

$\frac{dP}{dt} =$ instantaneous change in population/year

Assumption:

"The rate of growth of a population is proportional to the size of the population"

which is equivalent to

$$\frac{dP}{dt} = kP,$$

for some constant *k.*

2. Newton's Law of Cooling

$T(t)$ = temp. of the object at time t,

$\frac{dT}{dt}$ = rate of change of temp

T_s = constant temp. of surroundings

$T - T_s$ = temp. difference.

Assumption:

> *"The rate of cooling is proportional to the temperature difference between the object and its surroundings"*

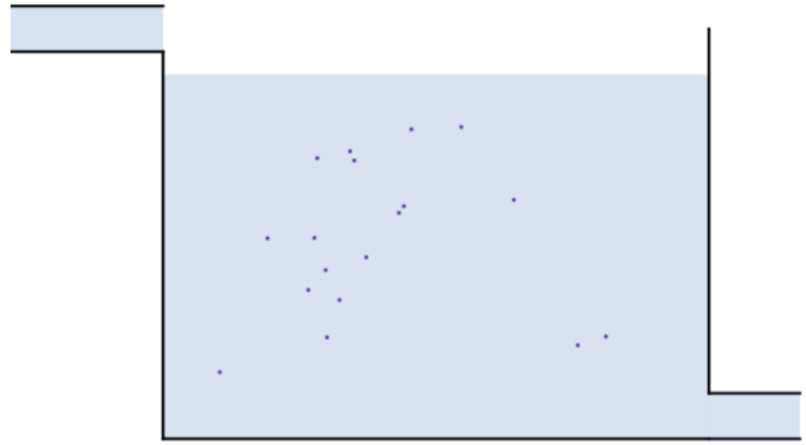
which is equivalent to

$$\frac{dT}{dt} = k(T - T_s),$$

for some constant k.

3. Mixing Problems

Assume a 50 gallon vat is initially full of pure water. A salt water mixture is being dumped **into** the vat at 2 gal/min and this mixture contains 3 grams of salt per gal. At the same time, the mixture is coming **out** of the vat at 2 gal/min.



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Let y(t) = grams of salt in vat at time *t*.

$\frac{y(t)}{50} =$ salt per gallon in vat at time, *t*.

$\frac{dy}{dt} =$ the rate (g/min) at which salt is changing with respect to time.

RATE IN = $\left(3\,\frac{\text{g}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = 6\,\frac{\text{g}}{\text{min}}$

RATE OUT = $\left(\frac{y}{50}\frac{\text{g}}{\text{gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right) = \frac{y}{25}\,\frac{\text{g}}{\text{min}}$

Thus,

$$\frac{dy}{dt} = 6 - \frac{y}{25}$$

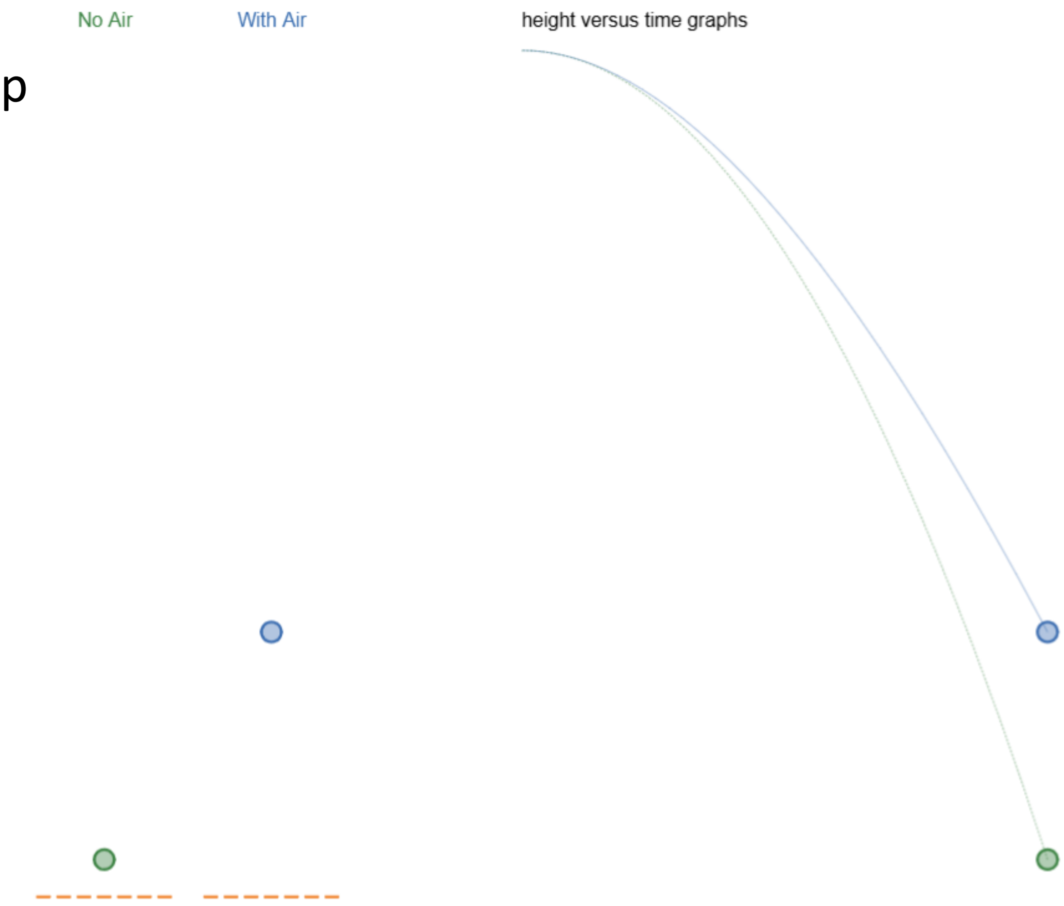
4. All motion problems!

Consider an object of mass *m* kg moving up and down on a straight line.

Let y(t) = `height at time *t*'

$$\frac{dy}{dt} = \text{`velocity at time } t\text{'}$$

$$\frac{d^2y}{dt^2} = \text{`acceleration at time } t\text{'}$$



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Newton's 2nd Law says:

$$m\frac{d^2y}{dt^2} = \text{sum of forces on the object}$$

No air

$$m\frac{d^2y}{dt^2} = -mg$$

With air

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}$$

5. Many, many others:

Example:

A common assumption for melting snow/ice is “the rate at which the object is melting (rate of change of volume) is proportional to the exposed surface area.”

Consider a melting snowball:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$$

Write down the differential equation for *r*.

My experiment with this problem…