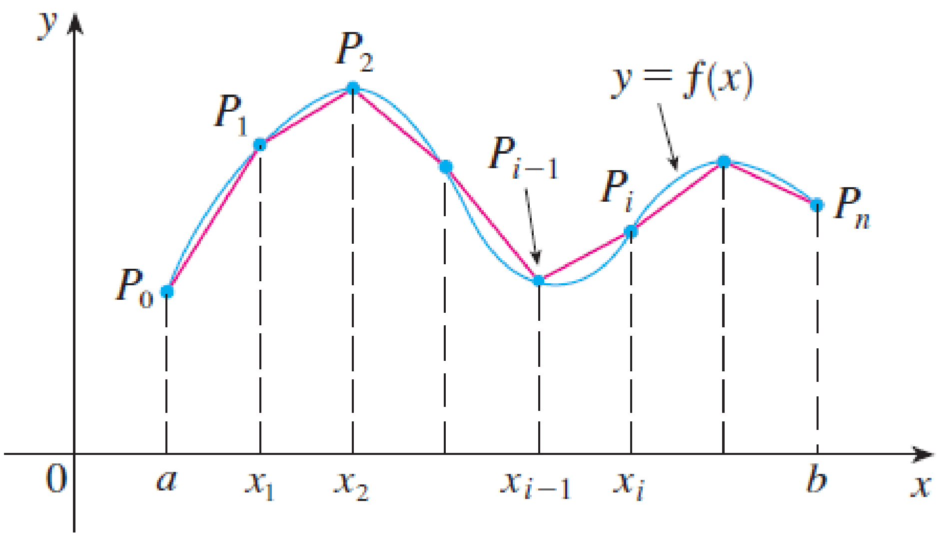
8.1 Arc Length

Goal: Given y = f(x) from x = a to x = b.

Want to find the ***length*** along the curve.



Break into *n* subdivision

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\Delta x$$

Compute $y_i = f(x_i)$.

Any ideas on how we can approximate the length?

Compute the straight-line distance from (x_i, y_i) to (x_{i+1}, y_{i+1}).

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

$$= \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y_i)^2}{(\Delta x)^2}\right)}$$

$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\ \Delta x$$

Add these distances up:

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\ \Delta x$$

Note:

$$\lim_{\Delta x\to 0} \frac{\Delta y_i}{\Delta x} = \text{slope of tangent} = f'(x)$$

Therefore:

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(f'(x)\right)^2}\ \Delta x$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Example:
Find the arc length of

$$y = 4x - 5 \quad \text{for} \quad -3 \le x \le 2.$$

Good news:
We have a method to write down an integral for arc length, yay!

Bad news:
The arc length integral almost always is something that can't be done explicitly with our methods, boo!

There are a few unusual functions where the integral can be computed exactly by using clever algebra. You will see several of these in homework and on the next page…

8.1 HW questions

Find the arc length of

1. $y = 4x - 5$ for $-3 \leq x \leq 2$.

2. $y = \sqrt{2 - x^2}$ for $0 \leq x \leq 1$.

3. $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ for $1 \leq x \leq 2$.

4. $y = \dfrac{1}{3}\sqrt{x}(x - 3)$ for $4 \leq x \leq 16$.

5. $y = \ln(\cos(x))$ for $0 \leq x \leq \pi/3$.

6. $y = \ln(1 - \mathrm{x}^2)$ for $0 \leq x \leq 1/7$.

Example: Find the arc length

$$y = \frac{x^4}{8} + \frac{1}{4x^2} \text{ for } 1 \le x \le 2$$

Example: Find the arc length

$$y = \ln(\cos(x)) \text{ for } 0 \leq x \leq \pi/3.$$

Arc Length is very important in motion (parametric) problems, which you will see a lot more of in Math 126:

$$x = x(t), y = y(t)$$

In this case, the same derivation from the beginning of class yields:

$$\text{Arc Length} = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

This gives the ***distance*** the object has traveled on the curve.

Arc Length (Distance) Function:

In motion problems we often use:

$$s(t) = \int_0^t \sqrt{\left(x'(u)\right)^2 + \left(y'(u)\right)^2}\, du$$

which gives the distance traveled from time 0 to time *t*. This is called the *Arc Length (Distance) Function*.

Simple Example:

Consider

$$x = 3t, y = 4t + 2$$

where *t* is in seconds.

(a) Find the arc length from 0 to 10 sec.

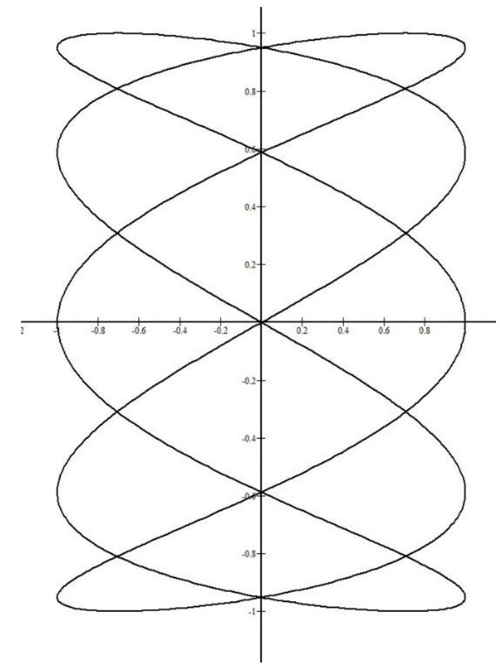
(b) Find the arc length function.

(c) What is the derivative of the arc length function?

(crazy looking) Old Exam Problem:

Consider the Lissajous curve given by the parametric equations

$$x = \cos(5t)\,, y = \sin(2t)$$



(a) Set up a definite integral for the arc length of the part of the curve for 0.10 ≤ t ≤ 0.25. DO NOT EVALUATE THE INTEGRAL.

(b) Use the Trapezoid Rule with n = 3 subintervals to approximate the definite integral in part (a).