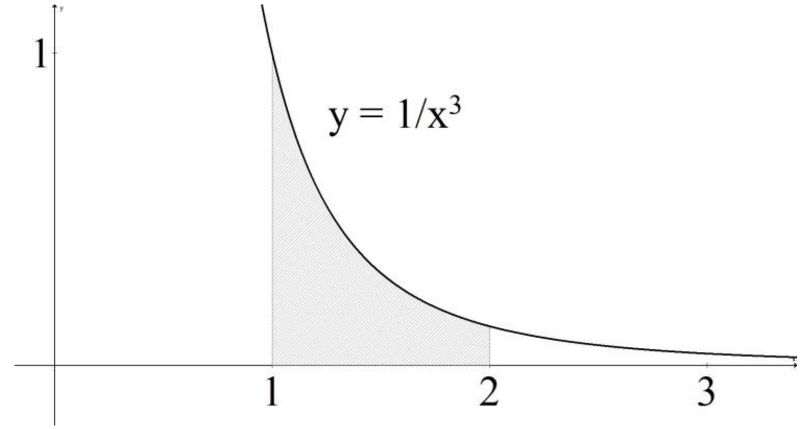
7.8 Improper Integrals

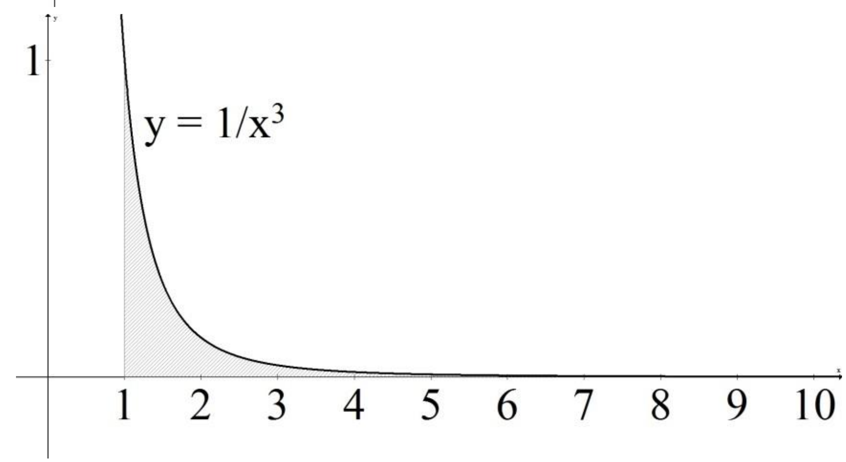
Motivation:

Consider the function $f(x) = \frac{1}{x^3}$.

Compute the area under the function…

1. …from $x = 1$ to $x = t$
2. …from $x = 1$ to $x = 10$
3. …from $x = 1$ to $x = 100$





Def'n: *Improper type 1 -*

$$\int_{a}^{\infty} f(x)dx = \lim_{t\to\infty} \int_{a}^{t} f(x)dx$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_{t}^{b} f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*. Otherwise, we say it *diverges*.

Example:

$$\int_{1}^{\infty} \frac{1}{x^3} dx =$$

Example:

$$\int_{-\infty}^{3} e^{2x} dx =$$

Example:

$$\int_{1}^{\infty} \frac{1}{x} dx =$$

Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

Def'n:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to-\infty} \int_{r}^{0} f(x)dx + \lim_{t\to\infty} \int_{0}^{t} f(x)dx$$

In this case, we say it *converges* only if both limits <u>separately</u> exist and are finite.

Def'n: *Improper type 2 -*

If $f(x)$ has a discontinuity at x = a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

If $f(x)$ has a discontinuity at x = b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*. Otherwise, we say it *diverges*.

Example:

$$\int_0^1 \frac{1}{\sqrt{x}}dx =$$

Example:

$$\int_0^2 \frac{x}{x-2}dx =$$

If $f(x)$ has a discontinuity at $x = c$ which is **between** a and b, then

$$\int_a^b f(x)dx = \lim_{r \to c^-} \int_a^r f(x)dx + \lim_{t \to c^+} \int_t^b f(x)dx$$

In this case, we say it *converges* only if both limits <u>separately</u> exist and are finite.

Example:

$$\int_0^\pi \frac{1}{\cos^2(x)}dx =$$

Limits Refresher

1. If stuck, plug in values "near" t.

2. Know your basic functions/values:

$$\lim_{t\to\infty} \frac{1}{t^a} = 0\,, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \frac{1}{e^{at}} = 0\,, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} t^a = \infty\,, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \ln(t) = \infty.$$

$$\lim_{t\to 0^+} \ln(t) = -\infty.$$

3. For indeterminant forms, use algebra and/or L'Hopital's rule

Examples:

$$\lim_{t\to 1} \frac{t^2 + 2t - 3}{t - 1} =$$

$$\lim_{t\to\infty} \frac{\ln(t)}{t} =$$

$$\lim_{t\to\infty} t^2 e^{-3t} =$$

A few general notes on **comparison**

Suppose you have two functions f(x) and g(x) such that $0 \leq g(x) \leq f(x)$ for all values.

- If $\int_1^\infty f(x)dx$ converges,
  then $\int_1^\infty g(x)dx$ converges.

- If $\int_1^\infty g(x)dx$ diverges,
  then $\int_1^\infty f(x)dx$ diverges.

You can verify that

$$\int_1^\infty \frac{1}{x^p}\,dx\,, \qquad \text{converges for } p > 1.$$

$$\int_1^\infty e^{px}\,dx\,, \qquad \text{converges for } p < 0.$$

And you can compare off of these to sometimes quickly tell if something else is converging or diverging (without calculating anything)!