7.3: Trig Substitution Review - Dr. Loveless

Concept: Use the following substitutions to simplify integrals with quadratics under a radical:

$$\sqrt{a^2-x^2} \Rightarrow x = a\sin(\theta) \qquad \sqrt{a^2+x^2} \Rightarrow x = a\tan(\theta) \qquad \sqrt{x^2-a^2} \Rightarrow x = a\sec(\theta)$$

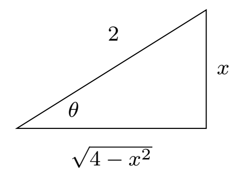
STEPS

1. Substitute for x and dx. Simplify using $1-\sin^2(\theta)=\cos^2(\theta)$, $1+\tan^2(\theta)=\sec^2(\theta)$, $\sec^2(\theta)-1=\tan^2(\theta)$.
2. Integrate using 7.2 techniques.
3. Convert back to x using a right triangle.

EXAMPLES

- **Example 1.** $\displaystyle\int \sqrt{4-x^2}\,dx$

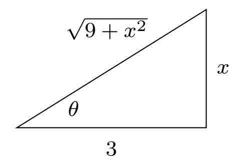
  1. Substituting $x=2\sin\theta$ and $dx=2\cos\theta\,d\theta$ simplifies to $\displaystyle\int\sqrt{4-x^2}\,dx = \int 4\cos^2\theta\,d\theta$.
  2. Using 7.2 methods: $\displaystyle\int 4\cos^2\theta\,d\theta = \cdots = 2\theta + 2\sin\theta\cos\theta + C$.
  3. The triangle gives $\sin\theta = \dfrac{x}{2}$, $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$.



Final Answer: $2\theta + 2\sin\theta\cos\theta + C = 2\sin^{-1}\left(\dfrac{x}{2}\right) + \dfrac{x\sqrt{4-x^2}}{2} + C.$

- **Example 2.** $\displaystyle\int \frac{x^3}{\sqrt{9+x^2}}\,dx$

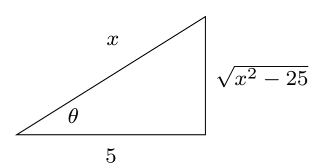
  1. Substituting $x=3\tan\theta$ and $dx=3\sec^2\theta\,d\theta$ simplifies to $\displaystyle\int \frac{x^3}{\sqrt{9+x^2}}\,dx = 9\int \tan^3\theta\,\sec\theta\,d\theta$.
  2. Using 7.2 methods: $\displaystyle 9\int \tan^3\theta\,\sec\theta\,d\theta = \cdots = 3\sec^3\theta - 9\sec\theta + C$.
  3. The triangle gives $\tan\theta = \dfrac{x}{3}$ and $\sec\theta = \dfrac{\sqrt{9+x^2}}{3}$.



Final Answer: $3\sec^3\theta - 9\sec\theta + C = 3\left(\dfrac{\sqrt{9+x^2}}{3}\right)^3 - 9\left(\dfrac{\sqrt{9+x^2}}{3}\right) + C.$

- **Example 3.** $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

  1. Substituting $x=5\sec\theta$ and $dx=5\sec\theta\tan\theta\,d\theta$ simplifies to $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx = 5\int \tan^2\theta\,d\theta$.
  2. Using 7.2 methods: $\displaystyle 5\int \tan^2\theta\,d\theta = \cdots = 5(\tan\theta - \theta) + C$.
  3. The triangle gives $\sec\theta = \dfrac{x}{5}$ and $\tan\theta = \dfrac{\sqrt{x^2-25}}{5}$.



Final Answer: $5(\tan\theta - \theta) + C = \sqrt{x^2-25} - 5\,\sec^{-1}\left(\dfrac{x}{5}\right) + C.$