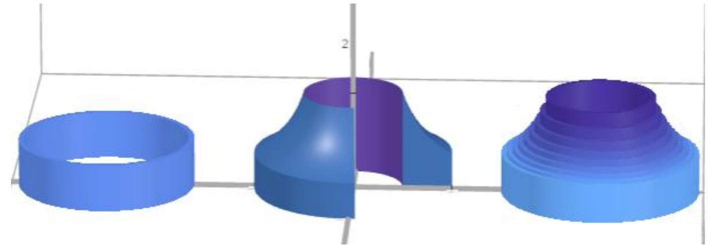
6.3 Volumes by Cylindrical Shells



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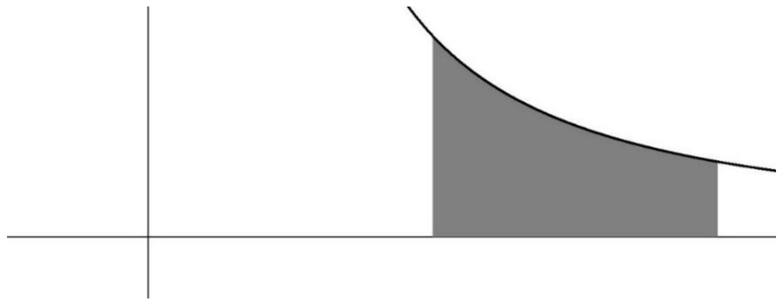
Motivational Example:

Let *R* be the region bounded by

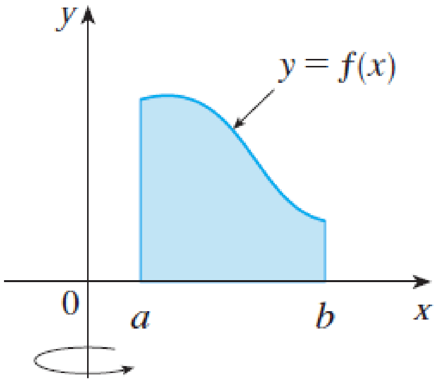
$$y = \frac{1}{x^2} + \frac{1}{x}, y = 0, x = 1, x = 2.$$

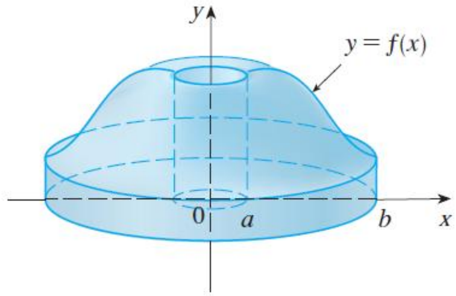
Consider the solid obtained by rotating about the **y-axis**.

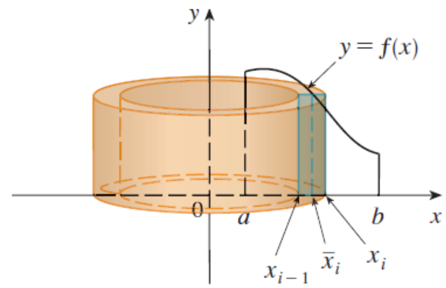


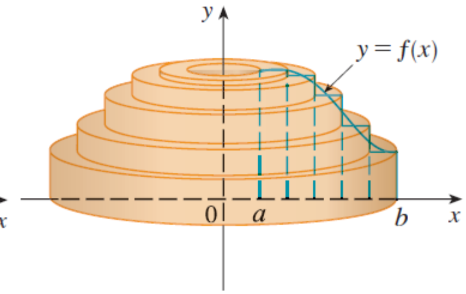
Try to use cross-sectional slicing… why is this difficult/messy?

Visual Motivation: Consider the solid









Derivation

Volume of one thin shell

≈ (surface area)(thickness)

= $SA(x_i)\ \Delta x$

= 2π(radius)(height)(thickness)

$$\text{Total Volume} \approx \sum_{i=1}^{n} \mathrm{SA}(\mathrm{x}_i)\ \Delta \mathrm{x}$$

$$\text{Exact Volume} = \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{SA}(\mathrm{x}_i)\Delta \mathrm{x}$$

Thus,

$$\text{Volume} = \int_a^b SA(x)dx$$

$$= \int_a^b 2\pi(radius)(height)dx$$

Volume using cylindrical shells

1. Draw region. "Cut" **parallel** to rotation axis.
   Label *x* if cut crosses the *x*-axis.
   Label *y* if cut crosses the *y*-axis.
   Label **everything** in terms this variable.

2. Surface area of cylindrical shell?
   SA = (Circumference)(Height)
   = 2π(Radius)(Height)
   Fill in with labels from graph.

3. Integrate the SA formula.

$$\text{Volume} = \int_a^b 2\pi(radius)(height)dx$$

Example:

Let R be the region bounded by

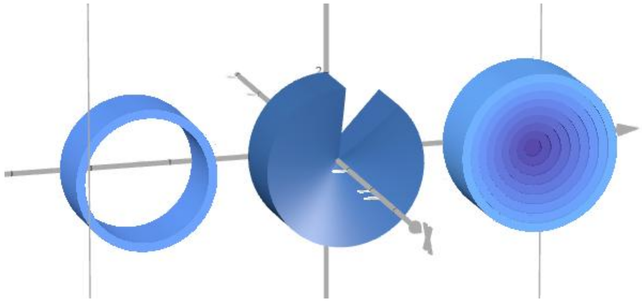
$$y = \frac{1}{x^2} + \frac{1}{x}, y = 0, x = 1, x = 2.$$

Set up an integral for the volume obtained by rotating R about the
***y*-axis**.

Example: Let R be the region in the first quadrant that is bounded by

$x = \sqrt{y + 1}$ and $y = 1$.

Find the volume obtained by rotating R about the ***x*-axis**.



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Flow chart of all Volume of Revolution Problems

***Step 0*: Draw an accurate picture!!! (Always draw a picture)**

Step 1: Choose and ***label*** the variable (based on the region and given equations)

If *x*, draw a typical **vertical** thin approximating rectangle at *x*.

If *y*, draw a typical **horizontal** thin approximating rectangle at *y*.

Step 2: Is the approximating rectangle *perpendicular* or *parallel* to the rotation axis?

Perpendicular → *Cross-sections*:

$$\text{Volume} = \int_a^b (\pi(\text{outer})^2 - \pi(\text{inner})^2)(\text{dx or dy})$$

Parallel → *Shells*:

$$\text{Volume} = \int_a^b 2\pi(\text{radius})(\text{height})(\text{dx or dy})$$

Step 3: Write everything in terms of the desired variable and fill in patterns.

Then integrate.

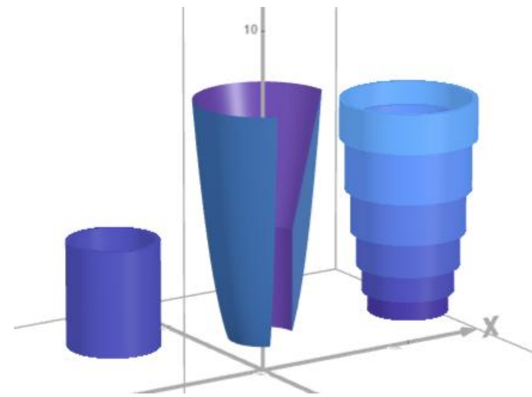
If you are still having trouble seeing which variable goes with which method here is a summary:

Axis of rotation	Disc/Washer	Shells
x-axis (or any horizontal axis)	dx	dy
y-axis (or any vertical axis)	dy	dx

Example 1:

Let R be the region bounded by $y = x^3$, $y = 4x$ between $x = 1$ and $x = 2$.

Set up an integral for the volume of the solid obtained by rotating R **about the y-axis**.



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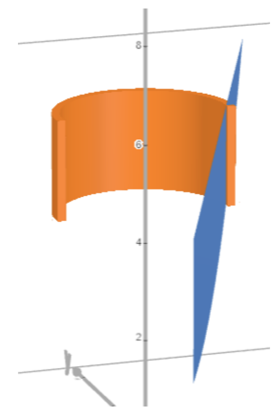
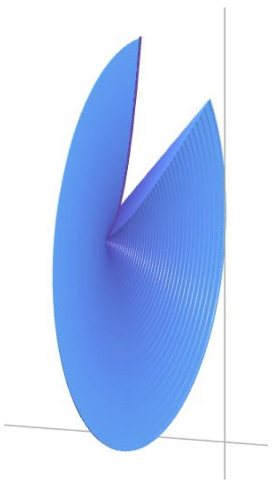


Example 2: Same region, different axis

Let R be the region bounded by $y = x^3$, $y = 4x$ between $x = 1$ and $x = 2$.

Set up an integral for the volume of the solid obtained by rotating R **about the x-axis**.



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Example 3: Same region again, different axis

Let R be the region bounded by $y = x^3$, $y = 4x$ between x = 1 and $x = 2$.

Set up an integral for the volume of the solid obtained by rotating R **about the vertical line x = 3**.

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