Overview of 3.9: Related Rates

Idea: In a given scenario, it is sometimes common that the rate of change of one quantity is known (or can be found). If we know the relationships between the quantities themselves, then a related rates question asks to find the relationships between the rates of the quantities. In other words, once we find one rate in a problem, how can we find the others? Depending on the scenario, there certainly can be some rates that are constants while others that depend on the values of the quantities themselves. We shouldn't be surprised by this. The key to these problems is to first find the general relationships between the rates (*before plugging in any values*), then evaluate this relationship with the particular quantities in question.

Recipe for Solving a Related Rates Problem:

Step 1: Draw a good picture. Label **everything**and give variable names to any changing quantities.

Step 2: Determine what information you **know** and what you **want** to find (and where).

Step 3: Find an equation relating the relevant variables. This usually involves a formula from geometry (inside the front cover of your text), similar triangles, the Pythagorean Theorem, or a formula from trigonometry. Use your picture!

Step 4: Use implicit differentiation to differentiate the equation with respect to time t.

Step 5: Substitute in what you **know** from Step 2 into **all** the relationships you have found and solve for the quantity you **want**. Do **not** substitute before this step!

A few tools

- Area of a circle $= \pi r^2$, Circumference of a circle $= 2\pi r$.

- Volume of a sphere $= \frac{4}{3}\pi r^3$, Surface area of a sphere $= 4\pi r^2$

- Volume of a cone $= \frac{1}{3}\pi r^2 h$, Volume of a cylinder $= \pi r^2 h$.

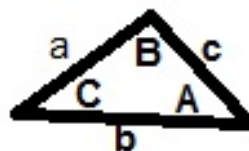- Similar Triangles: $\frac{a}{b} = \frac{d}{e}$ and $\frac{b}{c} = \frac{e}{f}$ and so on... for a situation such as:
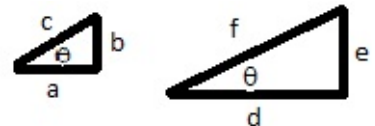
- Trig Definitions: $\tan(\theta) = \frac{b}{a}$, $\sin(\theta) = \frac{b}{c}$, $\cos(\theta) = \frac{b}{c}$ for the right triangle pictured above.

- Pythagorean Theorem: $a^2 + b^2 = c^2$ for **right** triangles as in picture above.

- Law of Cosines: $a^2 + b^2 - 2ab\cos(\theta) = c^2$ for all triangles as pictured here:
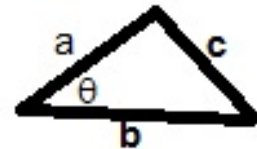
- Law of Sines: $\frac{b}{a} = \frac{\sin(B)}{\sin(A)}$ for all triangles as pictured here:

Examples:

0. The radius of a circle is increasing at a constant rate of 2 cm/min. At what rate is the area of the circle increasing when the radius is 10 cm?

1. Gas is escaping from a spherical balloon at a rate of 10 cubic feet per hour. At what rate is the radius changing when the volume is 400 cubic feet?

2. A man 7 feet tall is 20 feet from a 28-foot lamppost and is walking toward it at a rate of 4 feet per second. How fast is his shadow shrinking at that moment? How fast is the tip of the shadow moving?

3. A kite in the air at an altitude of 400 feet is being blown horizontally at the rate of 10 feet per second away from the person holding the kite string at ground level. At what rate is the string being let out when 500 feet of string is already out?

4. One bicycle is 4 miles east of an intersection, travelling toward the intersection at the rate of 9 miles per hour. At the same time, a second bike is 3 miles south of the intersection and is travelling away from the intersection at a rate of 10 miles per hour. At what rate is the distance between them changing? Is this distance increasing or decreasing?

5. A 13-foot ladder is leaning against a wall and its base is slipping away from the wall at a rate of 3 feet per second when it is 5 feet from the wall. How fast is the top of the ladder dropping at that moment?

6. A point moves on the graph of $y = x^3 + x^2 + 1$, the x-coordinate changing at a rate of 2 units per second. How fast is the y-coordinate changing at the point $(1, 3)$? How fast is the angle of inclination, θ, joining the point to the origin changing when x $= 1$?