Outline of Proof of $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$

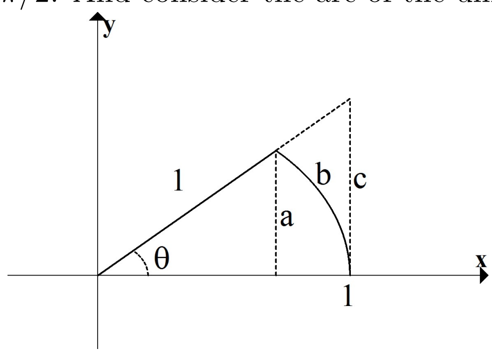
When we study slopes of trig functions, the limit $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$ often appears. My goal in this supplement is to give you a quick outline of the key elements of the proof that if θ is in radians, then this limit is equal to 1. You do not need to be able to prove this fact on exams, so this supplement is for your own interest.

First, remember that π is the number that you get when you take the circumference of a circle and divide by the diameter. Thus,

$$\text{circumference } = \pi(\text{diameter}) = \pi(2r) = 2\pi r.$$

Let's make a little note of what this means:
If you go all the way around a circle ($\theta = 2\pi$ rad), then the distance you travel is $2\pi r = \theta r$.
If you go half way around a circle ($\theta = \pi$ rad), then the distance you travel is $\pi r = \theta r$.
If you go 1/4 of the way around a circle ($\theta = \frac{\pi}{2}$ rad), then the distance you travel is $\frac{\pi}{2} r = \theta r$.
If we use radians, then Arc Length $= \theta r$. In fact, this is the main reason we use radians.

Now to the proof of the desired limit:
Let θ be an angle between 0 and $\pi/2$. And consider the arc of the unit circle that goes with this angle:



Note three things:

1. Looking at the small right triangle inside the arc, we get: $\sin(\theta) = \frac{a}{1}$, so $a = \sin(\theta)$.
2. Since Arc Length $= r\theta$ and $r = 1$, we get $b = \theta$ (only true in radians!!).
3. Looking at the large right triangle, we get: $\tan(\theta) = \frac{c}{1}$, so $c = \tan(\theta)$.

Key Observation: From the picture you can see: $a < b$ and $b < c$.
(For a more precise proof talk to me or see the appendix of the book).
Putting these facts together gives:

$$a < b \ \Rightarrow \ \sin(\theta) < \theta \ \Rightarrow \ \frac{\sin(\theta)}{\theta} < 1 \quad \text{and} \quad b < c \ \Rightarrow \ \theta < \tan(\theta) \ \Rightarrow \ \cos(\theta) < \frac{\sin(\theta)}{\theta}.$$

Therefore,

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1.$$

As $\theta \to 0$, both sides of these inequalities approach 1, so by the squeeze theorem we have:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$