Summary of Max/Min Facts

ORIGINAL ($f(x)$)	DERIVATIVE ($f'(x)$)	SECOND DERIVATIVE ($f''(x)$)
$f(x)$ = height of original at x	$f'(x)$ = slope of original at x	
increasing (uphill left-to-right)	positive (above x-axis)	
decreasing (downhill left-to-right)	negative (below x-axis)	
horizontal tangent	zero (crosses x-axis)	
concave up		positive
concave down		negative
possible inflection point		zero

Here is how we analyze critical points and points of inflection using calculus.

Step 1: What is the domain? (Is the domain given? Any places where the original function is not defined?).

Step 2: Find all the **critical numbers**.
We defined a **critical number** of $f(x)$ to be any number $x = c$ that is in the domain of $f(x)$ such that

1. $f'(c) = 0$, or
2. $f'(c)$ does not exist.

Note: $x = c$ has to be in the domain to be called a critical number.

Step 3a: If you are asked to find the absolute (global) maximum or minimum of $f(x)$ over a given domain.

1. Find the critical numbers.
2. Evaluate $f(x)$ at the critical numbers.
3. Evaluate $f(x)$ at the endpoints.
4. Conclusion:
   Biggest output = the absolute max and it occurs at the corresponding x value(s).
   Smallest output = the absolute min and it occurs at the corresponding x value(s).

Step 3b: If you are asked to classify the critical points as local max, local min or neither, then:

Option 1: **The First Derivative Test** If $x = c$ is a critical point of $f(x)$ and

1. If $f'(x)$ changes from negative to positive at $x = c$, then $x = c$ corresponds to a local minimum of f.
2. If $f'(x)$ changes from positive to negative at $x = c$, then $x = c$ corresponds to a local maximum of f.
3. If $f'(x)$ does not change at $x = c$, then $x = c$ is not a local max and not a local min of f.

Option 2: **The Second Derivative Test** If $x = c$ is a critical point of $f(x)$ and

1. If $f''(c) > 0$, then $x = c$ corresponds to a local minimum of f.
2. If $f''(c) < 0$, then $x = c$ corresponds to a local maximum of f.
3. If $f''(c) = 0$ or if $f''(c)$ does not exist, then the test is inconclusive
   (we can't conclude if $x = c$ is a local max/min or neither based on this information alone).

Step 4: **Points of inflection**.
A function is **concave up** at x if the tangent line is below the curve at x (so $f''(x) > 0$).
A function is **concave down** at x if the tangent line is above the curve at x (so $f''(x) < 0$).
Any point where the graph changes concavity is called a **point of inflection**.

Point of Inflection Test:
First solve for all places in the domain where $f''(x) = 0$ or $f''(x)$ does not exist.

If $x = c$ is a point where $f''(x) = 0$ or $f''(x)$ does not exist, then

if $f''(x)$ changes sign (pos-to-neg or neg-to-pos) at $x = c$, then $x = c$ gives a point of inflection.

At the end of a calculus course, every student should be able to quickly analyze any function and get the basic shape and features. Here are three basic examples:

1. Analyze the function $f(x) = x^4 - 2x^3$.

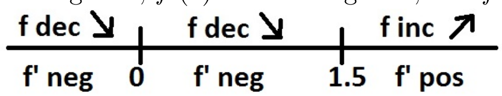
   First Derivative Facts:

   (a) *Critical Points*: First, we find $f'(x) = 4x^3 - 6x^2$. Solving gives

   $$\begin{aligned} 4x^3 - 6x^2 &= 0, && \text{factoring out } x^2 \text{ gives} \\ x^2(4x - 6) &= 0 && \text{simplifying gives} \\ x &= 0 \text{ or } 6/4 \end{aligned}$$

   We get two critical numbers $x = 0$ and $x = \frac{6}{4} = \frac{3}{2} = 1.5$.

   (b) *Number Line*: We plug in -1, 1, and 2 (or any values before 0, between 0 and 1.5, and after 1.5) to the derivative and find that $f'(-1) = -10$ is negative, $f'(1) = -2$ is negative, and $f'(2) = 8$ is positive. We summarize:

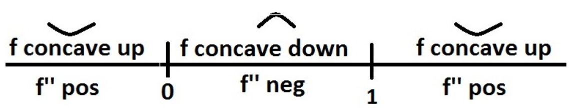
   Second Derivative Facts:

   (a) *Possible Points of Inflection*: First, we find $f''(x) = 12x^2 - 12x$. Solving gives:

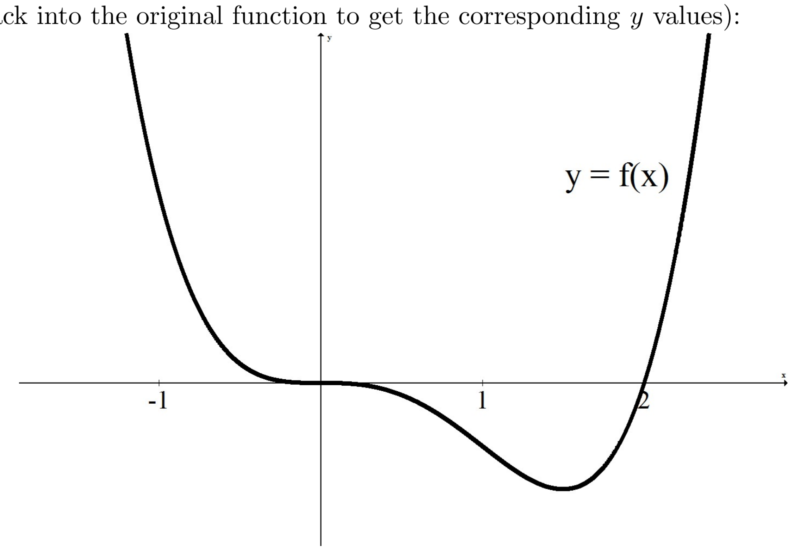
   $$\begin{aligned} 12x^2 - 12x &= 0, && \text{factoring out } 12x \text{ gives} \\ 12x(x - 1) &= 0 \end{aligned}$$

   We get two possible points of inflection $x = 0$ and $x = 1$.

   (b) *Number Line*: We plug in -1, 1/2 and 2 (or any values before 0, between 0 and 1, and after 1) to the second derivative and find that $f''(-1) = 24$ is positive, $f''(1/2) = -3$ is negative and $f''(2) = 24$ is positive. We summarize:

   Summary: $x = 1.5$ gives a local minimum, $x = 0$ and $x = 1$ give points of inflection (Note $x = 0$ is a horizontal point of inflection). And we should be able to roughly sketch the shape of the function (for a more accurate sketch, plug these values back into the original function to get the corresponding y values):

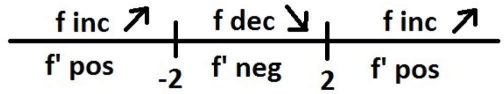
   

2. Analyze the function $f(x) = x^3 - 12x$.

First Derivative Facts:

(a) *Critical Points*: First, we find $f'(x) = 3x^2 - 12$. Solving $3x^2 - 12 = 0$ gives $x^2 = 4$. Thus, $x = -2$ and $x = 2$ are the critical numbers.

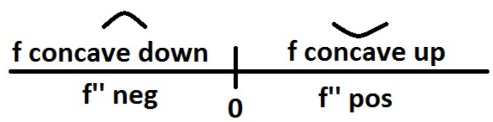
(b) *Number Line*: We plug in -3, 0, and 3 to the derivative and find that $f'(-3)$ is positive, $f'(0)$ is negative, and $f'(3)$ is positive. We summarize:



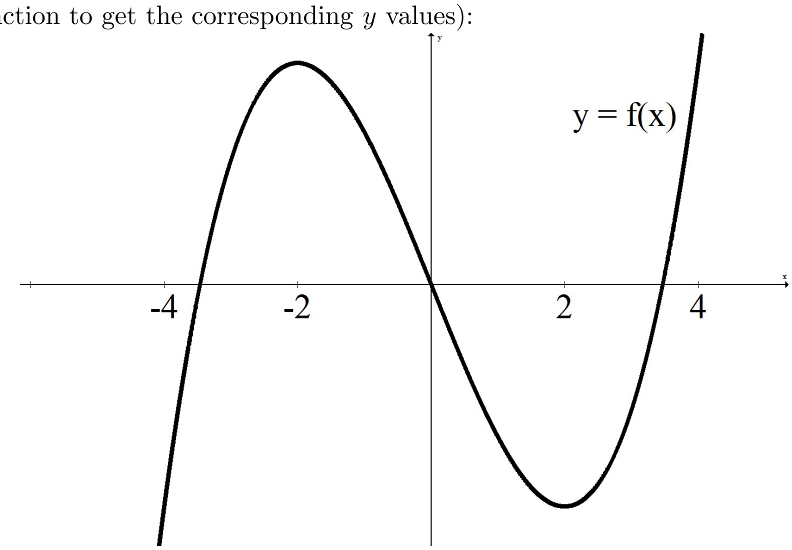
Second Derivative Facts:

(a) *Possible Points of Inflection*: First, we find $f''(x) = 6x$. Solving $6x = 0$ gives $x = 0$. Thus, $x = 0$ is the only possible point of inflection.

(b) *Number Line*: We plug in -1 and 1 to the second derivative and find that $f''(-1)$ is negative and $f''(1)$ is positive. We summarize:



Summary: $x = -2$ gives a local maximum, $x = 2$ gives a local minimum and $x = 0$ gives a point of inflection. And we should be able to roughly sketch the shape of the function (for a more accurate sketch, plug these values back into the original function to get the corresponding y values):



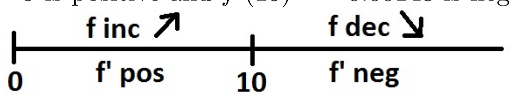
3. Analyze the function $f(x) = \frac{1}{x} - \frac{5}{x^2}$ for values $x > 0$.

First Derivative Facts:

(a) *Critical Points*: First, since $f(x) = x^{-1} - 5x^{-2}$, we have $f'(x) = -x^{-2} + 10x^{-3}$. We simplify to get $f'(x) = -\frac{1}{x^2} + \frac{10}{x^3}$ and try to solve:

$$\begin{aligned} -\tfrac{1}{x^2} + \tfrac{10}{x^3} &= 0, && \text{multiplying by } x^3 \text{ gives} \\ -x + 10 &= 0 && \text{simplifying gives} \\ x &= 10 \end{aligned}$$

(b) *Number Line*: We plug in 1 and 15 (anything between 0 and 10 and anything after 10) to the derivative and find that $f'(1) = -1 + 10 = 9$ is positive and $f'(15) \approx -0.00148$ is negative. We summarize:

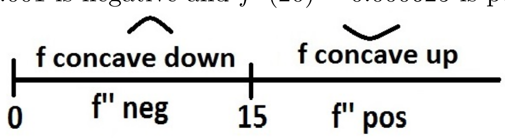


Second Derivative Facts:

(a) *Possible Points of Inflection*: First, we find $f''(x) = 2x^{-3} - 30x^{-4}$. We simplify to get $f''(x) = \frac{2}{x^3} - \frac{30}{x^4}$ and try to solve:

$$\begin{aligned} \tfrac{2}{x^3} - \tfrac{30}{x^4} &= 0, && \text{multiplying by } x^4 \text{ gives} \\ 2x - 30 &= 0 && \text{simplifying gives} \\ x &= 15 \end{aligned}$$

(b) *Number Line*: We plug in 10 and 20 (anything between 0 and 15 and anything after 15) to the second derivative and find that $f''(10) = -0.001$ is negative and $f''(20) = 0.000625$ is positive. We summarize:



Summary: $x = 10$ gives a local maximum and $x = 15$ gives a point of inflection. And we should be able to roughly sketch the shape of the function (for a more accurate sketch, plug these values back into the original function to get the corresponding y values):