Introduction to Parametric Equations

Typical, high school pre-calculus and algebra courses only discuss parametric equations lightly and focus on the fundamental functions (polynomials, exponentials, trig, *etc.*) and this is a perfectly reasonable approach. However, when it comes time to use our mathematical toolbox on real applied problems, parametric equations naturally arise. Particularly when the we encounter motion for which the location is a function of time. For many of these scenarios, it is easier and much more useful to have the coordinates x and y given as separate functions of time (and this will be even more useful as we go to 3 dimensions in Math 126). The basic facts we have so far (after our lecture on 10.1) are as follows:

1. If $x = x(t)$ and $y = y(t)$, then we can graph the parametric curve in the (x, y) plane by:
   - Selecting various values of t and calculating x and y. Then plot these points (x, y) and indicate the direction of increasing time.
   - Attempting to eliminate the parameter by either solving for t in one equation and using this to replace t in the other equation or using some identity (likely the trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$) to combine the equations and eliminate the parameter. If we in fact eliminate the parameter to get an equation in terms of x and y, then that equation represents the path of curve (but this equation doesn't contain any time information so we still have to go back to the parametric equations to plot some points and indicate direction).

The most common parametric equations we encounter this quarter are:

- Uniform Linear Motion (motion on a straight line at a constant speed):

  $$x(t) = a + bt \text{ and } y(t) = c + dt,$$

  where (a, c) is the 'initial', that is $t = 0$, (x, y) location and b and d are the horizontal and vertical velocities, respectively.

- Uniform Circular Motion around the origin (motion on a circle at a constant speed):

  $$x(t) = r\cos(\theta_0 + \omega t) \text{ and } y(t) = r\sin(\theta_0 + \omega t),$$

  where r is the radius of the circle, θ_0 is the angle corresponding to our initial position, and ω is the angular speed. Note: Standard angle measurement is always measured in radians in a counterclockwise fashion from the positive x-axis. And we want ω in units of rad/time (remember 2π radians = 1 revolution). We get elliptical motion if $x(t)$ and $y(t)$ have different values in the location for r (in which case there is no longer constant speed).

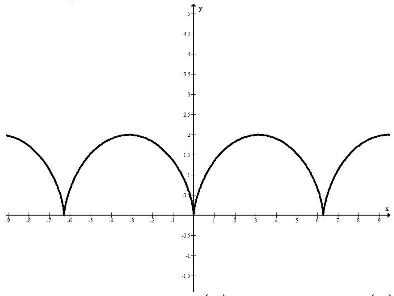
- Projectile Motion without air resistance:

  $$x(t) = x_0 + v_0\cos(A)t \text{ and } y(t) = y_0 + v_0\sin(A)t - \frac{g}{2}t^2,$$

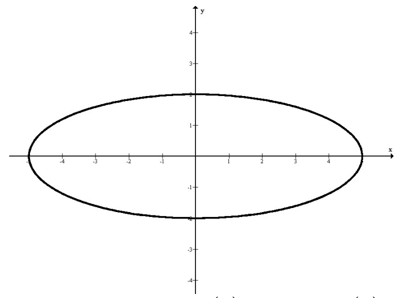
  where (x_0, y_0) is the initial location, v_0 is the initial velocity, A is the initial angle at which the projectile is fired, and g is the magnitude of the acceleration due to gravity.

A Bit About Parametric Equation Graphs

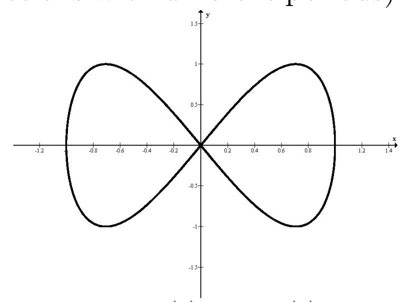
Recall: A **function**, $y = f(x)$, has only one y value for each x value. In other words, it passes the vertical line test. Function can look quite exotic, but they cannot have two y's for one x. I hope we have a basic understanding of what graphs of all the fundamental pre-calculus functions look like.
For parametric equations, the graphs (even from basic functions) can be even more quite exotic. Since x and y are given separately, the graph depends on the interplay between the functions defining x and y. Parametric curves often don't pass the vertical line test in the xy-plane, which also adds a new wrinkle to what the graphs can look like. In any case, if we can't quickly eliminate the parameter, then we typically will have to plot a lot of points to get a full idea of what the graph looks like, or we will have to do a careful analysis of what is happening in terms of x and y separately.
Here are some examples of more exotic parametric curves just to give you an idea of what can happen (especially in the case where x and y are defined in terms of sinusoidal functions with different periods).



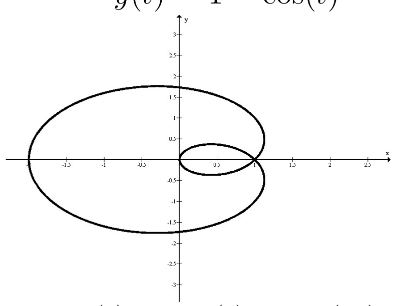
Cycloid: $x(t) = t - \sin(t)$
$y(t) = 1 - \cos(t)$



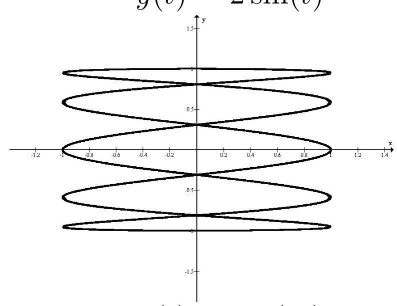
Ellipse: $x(t) = 5\cos(t)$
$y(t) = 2\sin(t)$



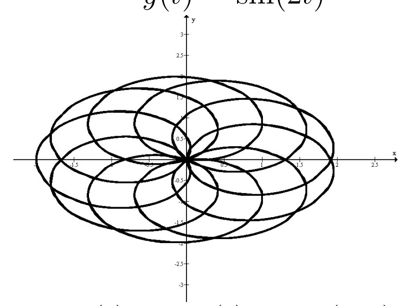
$x(t) = \cos(t)$
$y(t) = \sin(2t)$



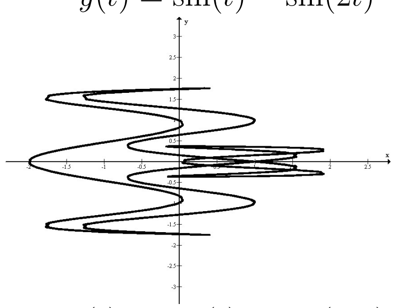
$x(t) = \cos(t) - \cos(2t)$
$y(t) = \sin(t) - \sin(2t)$



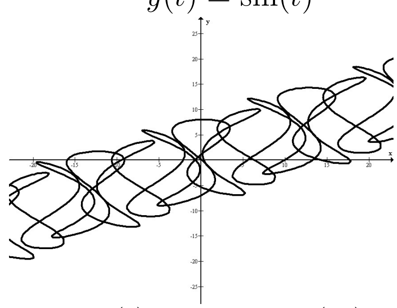
$x(t) = \cos(5t)$
$y(t) = \sin(t)$



$x(t) = \cos(t) - \cos(10t)$
$y(t) = \sin(t) - \sin(10t)$



$x(t) = \cos(t) - \cos(10t)$
$y(t) = \sin(t) - \sin(2t)$



$x(t) = 2t + 3\sin(7t)$
$y(t) = t + 8\cos(3t)$

It might be fun to use a graphing calculator, or online parametric grapher, to play around with different functions you can get. In graphics, parametric equations are useful as they are a good way to graph various pictures (and they are more flexible and easier to use than the typical $y = f(x)$ if you want to generate a particular picture).