Chapter 9: Inverse Functions

This document reviews core mechanics for finding inverse functions.

Key Concepts

- To find an inverse function, start with $y = f(x)$ and solve for x in terms of y. We call the resulting expression $x = f^{-1}(y)$.
- We often rewrite $f^{-1}(y)$ using x instead of y to express it as a function rule in terms of x, but this is a convention, not a requirement.
- Note that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$. You can use these to check your work.
- If a function is not one-to-one (i.e. does not pass the horizontal line test) on all $\mathbb{R}$ (e.g., a quadratic), restrict its domain to make it invertible. In this case, be clear about the domain of the inverse.

Example 1: Two Mechanical Inverse Problems

(a) Find $f^{-1}(y)$ for $y = f(x) = \frac{3}{4}x - \frac{5}{2}$.

Solution: Let $y = \frac{3}{4}x - \frac{5}{2}$. Solve for x in terms of y:

$$y = \frac{3}{4}x - \frac{5}{2} \quad \Rightarrow \quad y + \frac{5}{2} = \frac{3}{4}x \quad \Rightarrow \quad x = \frac{4}{3}\left(y + \frac{5}{2}\right) = \frac{4}{3}y + \frac{10}{3}.$$

Hence $f^{-1}(y) = \frac{4}{3}y + \frac{10}{3}$.

(b) Find $g^{-1}(y)$ for $y = g(x) = \sqrt[3]{2x+9}$.

Solution: Let $y = \sqrt[3]{2x+9}$. Solve for x in terms of y.

$$y = \sqrt[3]{2x+9} \quad \Rightarrow \quad y^3 = 2x + 9 \quad \Rightarrow \quad x = \frac{y^3 - 9}{2}.$$

Therefore $g^{-1}(y) = \frac{y^3 - 9}{2}$.

Example 2: Inverse of a Quadratic with a Restricted Domain

Find the inverse of $f(x) = 2(x-3)^2 + 5$ if $x \geq 3$. State the domain of f^{-1}.

Solution: Let $y = 2(x-3)^2 + 5$. Solve for x in terms of y:

From $y = 2(x-3)^2 + 5$: $\quad y - 5 = 2(x-3)^2 \quad \Rightarrow \quad \frac{y-5}{2} = (x-3)^2 \quad \Rightarrow \quad x - 3 = \sqrt{\frac{y-5}{2}}$ (since $x \geq 3$).

Thus $x = 3 + \sqrt{\frac{y-5}{2}}$.

$$f^{-1}(y) = 3 + \sqrt{\frac{y-5}{2}}, \qquad \text{domain of } f^{-1}:\ y \geq 5.$$