## Chapter 8: Function Composition

This document provides a concise review of Chapter 8, focusing on the mechanics, interpretation, and domain issues related to composition of functions.

### Key Concepts

- The composition of two functions is $(f \circ g)(x) = f(g(x))$.
- To compute $f(g(x))$, replace every occurrence of $x$ in $f(x)$ with $g(x)$.
- Notes on domain and range:
  - The domain of $f(g(x))$ is contained in the domain of $g(x)$ as the input to $g$ hasn't changed.
  - The range of $f(g(x))$ is contained in the range of $f(x)$ as the output is still from $f$.

On the following pages, we have compiled four examples to illustrate various key points:

1. Example 1: Basic Composition
2. Example 2: Domain and Range
3. Example 3: Working with multipart functions
4. Example 4: Working with absolute values

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### Example 1: Basic Composition of Functions

Let

$$f(x) = \sqrt{x+1} + 2, \qquad g(x) = 3x - 5.$$

Compute each of the following: $f(g(x))$, $g(f(x))$, $g(g(x))$, and $f(f(x))$.

*Solution:*

$$f(g(x)) = \sqrt{(3x-5)+1} + 2 = \sqrt{3x-4} + 2,$$

$$g(f(x)) = 3(\sqrt{x+1} + 2) - 5 = 3\sqrt{x+1} + 1,$$

$$g(g(x)) = 3(3x-5) - 5 = 9x - 20,$$

$$f(f(x)) = \sqrt{(\sqrt{x+1} + 2) + 1} + 2 = \sqrt{\sqrt{x+1} + 3} + 2.$$

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**Example 2: Composition and Domain/Range**

Let

$$f(x) = \sqrt{x+2}, \qquad g(x) = 2x - 4.$$

Since $\sqrt{x+2}$ is only defined if $x + 2 \geq 0$,

$$\boxed{\text{Domain of } f(x) : x \geq -2.}$$

We now explore how domain restrictions change under composition.

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(a) Find $f(g(x))$ and its domain.

*Solution:*

$$f(g(x)) = \sqrt{(2x-4)+2} = \sqrt{2x-2}.$$

The square root requires $2x - 2 \geq 0 \Rightarrow x \geq 1$.

$$\boxed{\text{Domain of } f(g(x)) : x \geq 1.}$$

The domain differs from $f(x)$ because the input to $f$ has changed.

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(b) Find $g(f(x))$ and its domain.

*Solution:*

$$g(f(x)) = 2(\sqrt{x+2}) - 4 = 2\sqrt{x+2} - 4.$$

Here the restriction comes from $f(x)$, requiring $x + 2 \geq 0 \Rightarrow x \geq -2$.

$$\boxed{\text{Domain of } g(f(x)) : x \geq -2.}$$

The domain is the same as $f(x)$ because the input to $f$ has not changed.

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**Example 3: Composition with Multipart Functions**

Let

$$f(x) = 3 - x, \qquad g(x) = \begin{cases} 6x^2, & x < 2, \\ x + 10, & x \geq 2. \end{cases}$$

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(a) Find $f(g(x))$

*Solution:* The input to the multipart function $g$ does not change, so we keep its original domain split at $x = 2$.

- For $x < 2$: $f(g(x)) = 3 - (6x^2) = 3 - 6x^2$.
- For $x \geq 2$: $f(g(x)) = 3 - (x + 10) = -x - 7$.

The result is

$$f(g(x)) = \begin{cases} 3 - 6x^2, & x < 2, \\ -x - 7, & x \geq 2. \end{cases}$$

This example shows how applying a simple function like $3 - x$ to a multipart input produces a new function that maintains the same breakpoints.

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(b) Find $g(f(x))$

*Solution:* Now the input to the multipart function $g$ changes because we substitute $f(x) = 3 - x$. We must determine which region of $g$ each $x$ value belongs to:

$$f(x) < 2 \Rightarrow 3 - x < 2 \Rightarrow x > 1, \quad f(x) \geq 2 \Rightarrow 3 - x \geq 2 \Rightarrow x \leq 1.$$

Thus:

- For $x > 1$: $g(f(x)) = 6(3 - x)^2 = 6(9 - 6x + x^2) = 6x^2 - 36x + 54$.
- For $x \leq 1$: $g(f(x)) = (3 - x) + 10 = 13 - x$.

$$g(f(x)) = \begin{cases} 13 - x, & x \leq 1, \\ 6x^2 - 36x + 54, & x > 1. \end{cases}$$

The branch points in this composition come from when $f(x)$ crosses 2, illustrating how the outer function's domain boundaries are influenced by the inner substitution.

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**Example 4: Additional Multipart Practice with $f(x) = |x|$**

Let $f(x) = |x|$. Recall that $f(x) = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$ Find the multipart function for each composition below, using the convention that the negative case is listed first.

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(a) $f(2x - 4)$

*Solution:*

$$f(2x-4) = \begin{cases} 4 - 2x, & x < 2, \\ 2x - 4, & x \geq 2. \end{cases}$$

Here the brances shifts to $x = 2$ because that's where the inner quantity $2x - 4$ changes sign.

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(b) $4f(x) + 1$

*Solution:*

$$4f(x) + 1 = \begin{cases} -4x + 1, & x < 0, \\ 4x + 1, & x \geq 0. \end{cases}$$

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(c) $f(f(x) - 2)$

*Solution:*

First compute the inner expression: $f(x) - 2 = \begin{cases} -x - 2, & x < 0, \\ x - 2, & x \geq 0. \end{cases}$

This output becomes the input to the outer $f$. The branching works like this:

$$\begin{array}{ll} x < 0 & \Rightarrow\ f(x) - 2 = -x - 2 \\ \quad \text{if } -x - 2 < 0\ (x > -2) & \Rightarrow\ f(f(x) - 2) = -(-x - 2) = x + 2 \\ \quad \text{if } -x - 2 \geq 0\ (x \leq -2) & \Rightarrow\ f(f(x) - 2) = -x - 2 \\ & \\ x \geq 0 & \Rightarrow\ f(x) - 2 = x - 2 \\ \quad \text{if } x - 2 < 0\ (0 \leq x < 2) & \Rightarrow\ f(f(x) - 2) = -(x - 2) = -x + 2 \\ \quad \text{if } x - 2 \geq 0\ (x \geq 2) & \Rightarrow\ f(f(x) - 2) = x - 2 \end{array}$$

Putting these together:

$$f(f(x) - 2) = \begin{cases} -x - 2, & x \leq -2, \\ x + 2, & -2 < x < 0, \\ -x + 2, & 0 \leq x < 2, \\ x - 2, & x \geq 2. \end{cases}$$

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