Chapter 7 Review: Optimization

Optimization problems ask us to find the maximum or minimum value of a quantity—such as the greatest area, largest profit, or smallest distance. In Precalculus, these problems usually lead to a **quadratic** model. Our goal is to express what we want to maximize or minimize as a quadratic function, and then use the vertex formula to find the answer.

This topic is a preview of one of the most important ideas in Calculus, where you will learn to find maxima and minima for many other kinds of functions (not just quadratics).

The Optimization Process

1. **Visualize and label:** Draw a clear picture and label all quantities with variables. Label lengths, angles, and other relevant information.
2. **What do we want?** Identify the quantity that should be made as large or as small as possible.
3. **What is given?** Write down all given facts and relationships.
4. **Find a quadratic formula.** Use the given relationships to write your desired quantity as a single-variable quadratic function.
5. **Use the vertex formula.** Apply $x = -\frac{b}{2a}$ to find where the maximum or minimum occurs, then substitute back to find the corresponding value and interpret.

Vertex facts: For a quadratic $y = ax^2 + bx + c$, the vertex occurs at $x = -\dfrac{b}{2a}$.

- If $a < 0$, the parabola opens downward and the vertex gives a **maximum**.
- If $a > 0$, the parabola opens upward and the vertex gives a **minimum**.

Formulas you might need

Revenue: $R =$ (price per unit)(quantity)

Area: $A =$ (length)(width)

Distance: $d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Here are three problems to try, with solutions to follow on the next pages.

Example 1: Maximum Revenue

Bill sells liters of juice using a sliding price scale. If he sells 4 liters, the price is \$7.50 per liter (total revenue of \$30). If he sells 8 liters, the price is \$6.75 per liter (total revenue of \$54). The price per liter decreases linearly with liters sold. Find the quantity and price that will lead to the maximum revenue on one sale.

Example 2: Two Square Enclosures with Partitions

You have 280 meters of fencing to make *two* enclosures. Both will be squares; one of them has three interior partitions (dividing the square into four spaces). Find the dimensions that will minimize the total area.

Example 3: Closest Distance

Sven starts walking south at 6 ft/s from a point 130 ft north of an intersection. At the same time, Rudyard starts walking east at 4 ft/s from a point 190 ft west of the intersection. At what time are they closest together?

Example 1: Maximum Revenue

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Solution

Visualize/Label:
Let x = liters sold and $f(x)$ = total revenue in dollars.

What do we want?
The maximum total revenue, $f(x)$.

What is given?

The price $p(x)$ is a linear function of x liters sold.

$$p(x) = mx + b$$

Using the two data points $(4, 7.50)$ and $(8, 6.75)$,

$$m = \frac{6.75 - 7.50}{8 - 4} = -0.1875, \qquad b = 7.50 - (-0.1875)(4) = 8.25$$

so $p(x) = -0.1875x + 8.25$.

Find a quadratic function for revenue:

$$f(x) = x \cdot p(x) = x(-0.1875x + 8.25) = -0.1875x^2 + 8.25x$$

Use vertex formula:

$$x = -\frac{b}{2a} = -\frac{8.25}{2(-0.1875)} = 22 \text{ liters.}$$

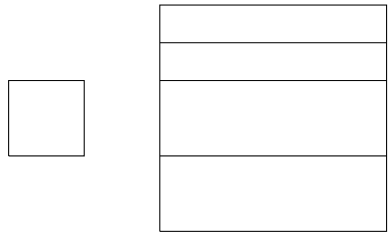
Since a is negative, the parabola opens downward and this gives a maximum.

$$f(22) = -0.1875(22)^2 + 8.25(22) = 90.75$$

Interpretation: If Bill sells 22 liters of juice it will maximize his revenue and give him \$90.75.

Example 2: Two Square Enclosures with Partitions

You have 280 meters of fencing to make *two* enclosures. Both will be squares; one of them has three interior partitions (dividing the square into four spaces). Find the dimensions that will minimize the total area.



Solution

Visualize/Label: Let x be the side (meters) of the simple square (no partitions). Let y be the side of the square that has three full-length partitions.

What do we want?
We want to minimize the total area $A = x^2 + y^2$.

What is given?
Fencing used = perimeter of simple square + (perimeter of partitioned square + three partitions):

$$4x + (4y + 3y) = 4x + 7y = 280.$$

Find a quadratic function (in one variable).

Solve for y:

$$y = \frac{280 - 4x}{7}.$$

Substitute into $A = x^2 + y^2$:

$$A(x) = x^2 + \left(\tfrac{280-4x}{7}\right)^2 = x^2 + \frac{(280 - 4x)^2}{49} = x^2 + \frac{16x^2 - 2240x + 78400}{49}.$$

Combine like terms:

$$A(x) = \frac{65}{49}x^2 - \frac{2240}{49}x + 1600.$$

This is a quadratic in x with $a = \frac{65}{49} > 0$, so it opens upward.

Use vertex formula (minimum):

$$x = -\frac{-2240/49}{2(65/49)} = \frac{2240}{130} = \frac{224}{13} \approx 17.23 \text{ m.}$$

$$y = \frac{280 - 4x_{\min}}{7} = \frac{392}{13} \approx 30.15 \text{ m.}$$

$$A = x^2 + y^2 \approx 1205.1 \text{ m}^2.$$

Interpretation: To *minimize* the total area, split the fencing as found above.

Example 3: Closest Distance — Solution

Sven starts walking south at 6 ft/s from a point 130 ft north of an intersection. At the same time, Rudyard starts walking east at 4 ft/s from a point 190 ft west of the intersection. At what time are they closest together?

Solution

Visualize/Label:
Let t represent time in seconds and $d(t)$ represent the distance between Sven and Rudyard.

What do we want?
The minimum distance d between them after t seconds.

What is given?
Sven's position: $(0, 130 - 6t)$
Rudyard's position: $(-190 + 4t, 0)$

Find a quadratic function:

$$d(t) = \sqrt{(190 - 4t)^2 + (6t)^2} = \sqrt{52t^2 - 1520t + 36100}.$$

Since the square root is increasing, minimize the inside:

$$D(t)^2 = 52t^2 - 1520t + 36100.$$

Use vertex formula:

$$t = -\frac{-1520}{2(52)} = 14.615 \text{ seconds}.$$

Find minimum distance:

$$d_{\text{min}} = \sqrt{52(14.615)^2 - 1520(14.615) + 36100} \approx 85.98 \text{ ft}.$$

Interpretation: After about 14.6 seconds, the two are closest—about 86 feet apart.

Just for Fun – A Preview of Calculus 1 Optimization

All of the following problems come from our Calculus 1 course. You do **not** need to know how to solve these yet—they are here for you to get a feel for the kinds of optimization problems you will see later. Some can already be solved with quadratics, while others require new tools from Calculus. They are also just plain fun to look at!

CLASSIC OPTIMIZATION PROBLEMS (HOW TO SET UP)

1. *Basic finding numbers examples*:

(A) Homework Question: Find two numbers whose difference is 188 and whose **product is a minimum**.

(B) Old Final Question: The product of two positive numbers is 100. How **small can the sum of one of the numbers plus the square of the other number be**?

2. *Optimizing Area Questions*:

(A) Homework Question: Find the **area of the largest** rectangle that can be inscribed in a right triangle with legs of length 4 cm and 6 cm if two sides of the rectangle lie along the legs.

(B) Old Final Question: A farmer has 136 meters of fencing. She wants to make two rectangular enclosures. One will be a square. The other will have its long side twice as long as its short side (All the possibility that all of the fencing could go to only one of the enclosures.
What dimensions will make the combined total **area as small as possible**?
What dimensions will make the combined total **area as big as possible**?

3. *Minimizing cost/surface area/material used*:

(A) Homework Question: A box with a rectangular base and open top must have a volume of 10 cm^3. The length of one side of the rectangle is twice the width. The material for the base costs \$5.00 per square meter and the material for the sides costs \$3.00 per square meter. Find dimensions and the corresponding **cost for the cheapest container**.

(B) Old Final Question: An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to the storage tanks located on the south bank of the river 6km east of the refinery. The cost of laying pipe is \$300,000/km over land and \$500,000/km under the river. How should you lay the pipe to **minimize cost**? (A picture was included).

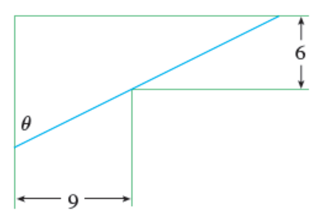
4. *General xy-plane optimization*:

(A) Homework Question: Find the point on the line y = 4x+3 that is **closest to the origin**.

(B) Old Final Question: Find the equation of the line through the point (3,5) that cuts off the **least area from the first quadrant**.

5. *Hints on Two Other Problems from Homework*:

(A) "Find the **longest steel pipe** that will fit around this corner."
Hint: This is the same as asking for the **<u>minimum length</u>** of this blue line that would touch both walls as shown,
because if it's longer than this minimum it will get stuck.



(B) "A woman at a point *A* on the shore of a circular lake with radius 2 mi wants to arrive at the point *C* diametrically opposite *A* on the other side of the lake in the shortest possible time (see the figure). She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. For what value of the angle θ shown in the figure will she **minimize her travel time**?"

Hints: Label the center as P. Draw a line segment connecting P and B. Note: this line segment has length 2. Also note, the triangle APB is isosceles, so the angle at B in this triangle is also θ which makes the angle in this triangle 180 - θ – θ = 180 - 2 θ. From this you can conclude that the angle BPC is 2 θ.
You can also assume that the angle ABC is a right angle (because it always will be).
And don't forget that the arc length around a circle is Arc Length = r θ (in radians).