Chapters 5: Functions and Functional Notation

This document provides a quick summary and review for Chapters 5.

Key Concepts

- Understand the definition of a **function** as well and the meaning of the **domain** and **range**.
- Understand the vertical line test.
- Be able to use functional notation: $f(2x)$, $f(x+h)$, $f(x+h)-f(x)$, etc.
- Understand how to use the difference quotient:

$$\frac{f(x+h)-f(x)}{h}.$$

Example: Difference Quotient Practice

For each function, compute and simplify $\dfrac{f(x+h)-f(x)}{h}$.
Calculus Note: This expression is the slope of the secant line from x to $x+h$. As $h \to 0$, the secant approaches the tangent line. Simplifying the difference quotient and then letting $h = 0$ gives the *derivative*, the slope of the tangent line at x. Mastering this algebra is essential, since it is the foundation of calculus. Here is a video of me using this definition to model the velocity of a tennis ball: Watch video

Example 1: If $f(x) = 4x+3$, then $\dfrac{f(x+h)-f(x)}{h} = \dfrac{[4(x+h)+3]-[4x+3]}{h}$.

- *Solution* Expanding and simplifying

$$\begin{aligned}
\tfrac{[4(x+h)+3]-[4x+3]}{h} &= \tfrac{4x+4h+3-(4x+3)}{h} && \text{distribute the 4}\\
&= \tfrac{4x+4h+3-4x-3}{h} && \text{distribute the negative}\\
&= \tfrac{4h}{h} && \text{cancel in the numerator}\\
&= 4 && \text{cancel } h
\end{aligned}$$

- Note how the slope of the 'tangent' line is 4 which makes since as this was just a line with slope 4.

Example 2: If $f(x) = 4x^2-3$, then $\dfrac{f(x+h)-f(x)}{h} = \dfrac{[4(x+h)^2-3]-[4x^2-3]}{h}$.

- *Solution* Expanding and simplifying

$$\begin{aligned}
\tfrac{[4(x+h)^2-3]-[4x^2-3]}{h} &= \tfrac{[4(x^2+2xh+h^2)-3]-[4x^2-3]}{h} && \text{expand the square}\\
&= \tfrac{[4x^2+8xh+4h^2-3]-[4x^2-3]}{h} && \text{distribute the 4}\\
&= \tfrac{4x^2+8xh+4h^2-3-4x^2+3}{h} && \text{distribute the negative}\\
&= \tfrac{8xh+4h^2}{h} && \text{cancel in the numerator}\\
&= \tfrac{8xh}{h} + \tfrac{4h^2}{h} && \text{distribute}\\
&= 8x+4h && \text{cancel}
\end{aligned}$$

- In calculus, we will say the slope of the tangent line is $f'(x) = 8x$, which is what we get when we plug in $h = 0$.

Example 3: If $f(x) = 2x^2-3x$, then 'YOU TRY'. Solution on next page.
Example 4: If $f(x) = \frac{1}{x}$, then 'YOU TRY'. Solution on next page.
Example 5: If $f(x) = \sqrt{x}$, then 'YOU TRY'. Solution on next page.

Example 3: If $f(x) = 2x^2 - 3x$, then $\dfrac{f(x+h)-f(x)}{h} = \dfrac{[2(x+h)^2 - 3(x+h)] - [2x^2 - 3x]}{h}$.

- *Solution* Expanding and simplifying

$$\begin{aligned}
\tfrac{[2(x+h)^2-3(x+h)]-[2x^2-3x]}{h} &= \tfrac{[2(x^2+2xh+h^2)-3x-3h]-[2x^2-3x]}{h} && \text{expand squares, distribute } -3 \\
&= \tfrac{[2x^2+4xh+2h^2-3x-3h]-[2x^2-3x]}{h} && \text{distribute the 2} \\
&= \tfrac{2x^2+4xh+2h^2-3x-3h-2x^2+3x}{h} && \text{distribute the negative} \\
&= \tfrac{4xh+2h^2-3h}{h} && \text{cancel in the numerator} \\
&= \tfrac{h(4x+2h-3)}{h} && \text{factor } h \\
&= 4x+2h-3 && \text{cancel } h
\end{aligned}$$

- In calculus, we will say the slope of the tangent line is $f'(x) = 4x - 3$, which is what we get when we plug in $h = 0$.

Example 4: If $f(x) = \frac{1}{x}$, then $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$.

- *Solution* Expanding and simplifying

$$\begin{aligned}
\tfrac{\frac{1}{x+h}-\frac{1}{x}}{h} &= \tfrac{\frac{x-(x+h)}{x(x+h)}}{h} && \text{combine into single fraction} \\
&= \tfrac{\frac{-h}{x(x+h)}}{h} && \text{simplify numerator} \\
&= \tfrac{-h}{h\cdot x(x+h)} && \text{rewrite denominator} \\
&= \tfrac{-1}{x(x+h)} && \text{cancel } h
\end{aligned}$$

- In calculus, we will say the slope of the tangent line is $f'(x) = \frac{-1}{x^2}$, which is what we get when we plug in $h = 0$.

Example 5: If $f(x) = \sqrt{x}$, then $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$.

- *Solution* Expanding and simplifying

$$\begin{aligned}
\tfrac{\sqrt{x+h}-\sqrt{x}}{h} &= \tfrac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} && \text{multiply by conjugate} \\
&= \tfrac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} && \text{difference of squares} \\
&= \tfrac{h}{h(\sqrt{x+h}+\sqrt{x})} && \text{cancel in numerator} \\
&= \tfrac{1}{\sqrt{x+h}+\sqrt{x}} && \text{cancel } h
\end{aligned}$$

- In calculus, we will say the slope of the tangent line is $f'(x) = \frac{1}{2\sqrt{x}}$, which is what we get when we plug in $h = 0$.

Observation: After simplification, plugging in $h = 0$ is safe. Again in the second week of calculus this becomes the definition of the derivative. And we use this to build a collection of shortcuts that you use for the rest of the calculus courses (and avoids having to do this algebra every time).