Chapter 4 Summary: Linear Modeling

This document provides a quick summary and review for Chapter 4.

Lines

Key Facts

Given two points on a non-vertical line (x_1, y_1) and (x_2, y_2), **slope**: $m = \frac{y_2 - y_1}{x_2 - x_1}$

- 'Rise over Run' Double-Check: Are y-coordinates on top? Do the coordinates 'line up'?.

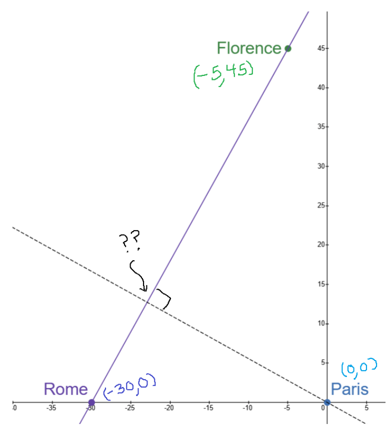
Any other point (x, y) on the line must satisfy the condition: $\frac{y - y_1}{x - x_1} = m$ which can also be written $y - y_1 = m(x - x_1)$.
Standard forms:

- Point-slope: $y = m(x - x_1) + y_1$
- Slope-intercept: $y = mx + b$

Tips

- Parallel lines have the same slope.
- Perpendicular lines have negative reciprocal slopes, meaning $m_2 = -\frac{1}{m_1}$.
- We can use perpendicular lines to answer many important questions including...
  1. Finding the location on a linear path closest to a given point. You can find the equation for the perpendicular line through the given point, then find the intersection of the two lines.
  2. Tangent lines to circles. The 'radial' line and the 'tangent' lines are perpendicular at a point on the circle. You will use this a lot in the first week of Math 124.

Example: Closest Point on a Line (Problem 4.10)



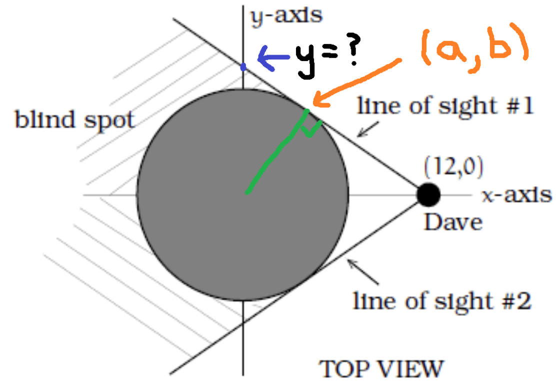
Pam is taking a train from the town of Rome to the town of Florence. Rome is located 30 miles due West of the town of Paris. Florence is 25 miles East, and 45 miles North of Rome (this is shown at right with Paris at the origin). Find the point on Pam's path when she is closest to Paris.

Solutions Outline

- Visualize - Impose a coordinate system and label the cities.
- Equations - Line thru Rome and Florence
  - $m = \frac{45-0}{-5-(-30)} = \frac{45}{25} = \frac{9}{5}$, so $y = \frac{9}{5}(x - (-30)) + 0$ or simply $y = \frac{9}{5}(x + 30)$.
- Translates - Find intersection with the perpendicular line thru Paris.
  - Perpendicular line through Paris has a negative reciprocal slope of the line through Rome and Florence, so slope $m = -\frac{5}{9}$ giving $y = -\frac{5}{9}(x - 0) + 0$, or simply $y = -\frac{5}{9}x$.
- Solve - Find intersection
  - Clear denominator: Multiplying both sides of $-\frac{5}{9}x = \frac{9}{5}(x + 30)$ by 45 (i.e. 5 and 9) gives...
  - $-25x = 81(x + 30)$, so$-25x = 81x + 2430$.
  - Thus, $-106x = 2430$, so $x = -\frac{2430}{106} \approx -22.9245$
  - And $y = -\frac{5}{9}x = -\frac{5}{9} \cdot -\frac{2430}{106} \approx 12.7358$
  - Point is $(-22.9245, 12.7358)$, we did it!

Example: Line of Sight (Problem 4.8)

Viewed from above, the image shows Dave standing next to a grain silo. The silo has radius 9 feet and Dave is standing 12 feet from the center of silo. Dave's vision of the y-axis is obstructed by the silo. Can you find the 'point of tangency' where Dave's line of sight touches the end of the silo?



Circle and Tangent Fact: The key observation is that the 'radial' line from the origin to the tangent point is perpendicular to the 'tangent' line.

Solutions Outline

- Visualize - Label the unknown point tangent point (a, b).
- Equations -
  - Note that (a, b) is on the circle so $a^2 + b^2 = 9^2$.
  - The slope from (a, b) to $(12, 0)$ is $m_1 = \frac{b-0}{a-12}$
  - The slope from $(0, 0)$ to (a, b) is $m_2 = \frac{b-0}{a-0}$
- Translates - The radial line and tangent line are perpendicular! So they have negative reciprocal slopes...
  - Thus, $m_1 = -\frac{1}{m_2}$, which means...
  - $\frac{b}{a-12} = -\frac{a}{b}$, thus, $b^2 = -a(a - 12)$, which simplifies to
  - $b^2 = -a^2 + 12a$ which is the same as $a^2 + b^2 = 12a$.
- Solve - We now have two facts about (a, b)
  - (a, b) is on the circle, so $a^2 + b^2 = 9^2$ and $a^2 + b^2 = 12a$
  - Combining gives $12a = 81$, so $a = \frac{81}{12} = \frac{27}{4} = 6.75$
  - From this, you can find b
    - $(27/4)^2 + b^2 = 9^2$, so $b = \sqrt{81 - (27/4)^2} \approx 5.9529$
  - Point of tangency is $(6.75, 5.9529)$, got it!

Important Note: Your first couple homework problems in Math 124 will look like this, so make a note of this process. It is how we introduce you to tangent lines in Math 124.

Uniform Linear Motion

Key Facts

Parametric equations for motion along a line at constant speed are given by:

$$x = x_1 + v_x t, \quad y = y_1 + v_y t$$

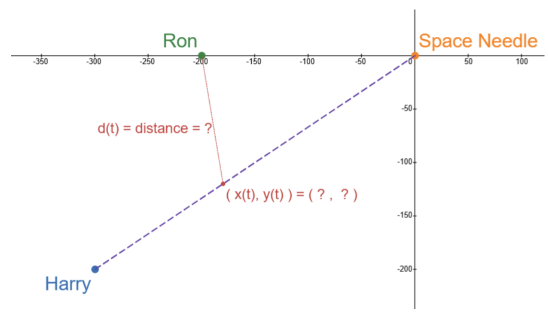
- (x_1, y_1) is the initial position at $t = 0$
- (x_2, y_2) is a known location the object reaches at time $t = T$.
- $v_x = \frac{x_2 - x_1}{T}$, $v_y = \frac{y_2 - y_1}{T}$ and are sometimes called the horizontal and vertical velocities.

Tips

- This is a useful additional tool to help you study objects moving at a constant speed on a straight line.
- This explicitly gives x and y in terms of time.
- Later in the Math 120 we will talk about uniform circular motion which is another situation where we use 'parametric equations'.
- In Math 124, you will talk a lot more about 'parametric equations'.
- So our goal here is to give you a small taste of these ideas now so that Math 124 isn't your first experience with parametric equations.

Example: Closest Point and Parametric Equations (Problem 4.13)

Ron is standing 200 feet due west of the space needle waiting for Harry. Harry is 200 feet south and 300 feet west of the space needle, then walks toward the space needle getting there in 5 seconds. (a) Find parametric equations for Harry's location at time t seconds. (b) Find a formula for the distance from Harry to Ron after t seconds.



Solutions Outline

- Visualize - Impose a coordinate system and label.
- Equations -
  - Harry goes from $(-300, -200)$ to $(0, 0)$ in 5 seconds, so
  - $(x_1, y_1) = (-300, -200)$ and $(x_2, y_2) = (0, 0)$ and $T = 5$, which gives
  - $v_x = \frac{0-(-300)}{5} = 60$ and $v_y = \frac{0-(-200)}{5} = 40$
  - $x(t) = -300 + 60t$, $y(t) = -200 + 40t$
- Translates - Want parametric equations, done. Now need to find distance.
  - Distance from (-200,0) to $(-300 + 60t, -200 + 40t)$ is given by

$$\sqrt{(-300 + 60t - (-200))^2 + (-200 + 40t - 0)^2} = \sqrt{(60t - 100)^2 + (40t - 200)^2}$$

- Solve - Wasn't asked to solve for anything, but perhaps a challenge is to find how long it would take for Harry to get to the point that is closest to Ron. Try it.