Chapter 14 Review Sheet: Linear-to-Linear Rational Models

Key Facts / What You Need to Know

- A **linear-to-linear rational model** has the form $f(x) = \dfrac{ax+b}{x+c}$, where a, b, c are constants.
- **Vertical asymptote:** denominator $= 0 \implies x = -c$.
- **Horizontal asymptote:** for large x, $f(x) \approx a$, so $y = a$.
- Once you draw $x = -c$ and $y = a$, the graph looks like a shifted/scaled version of $\frac{1}{x}$ or $-\frac{1}{x}$ (two opposite branches).
- Two main situations
  1. **Three points** $\rightarrow$ solve directly for a, b, c.
  2. **Two points + one asymptote** $\rightarrow$ use the asymptote first, then solve for the rest.

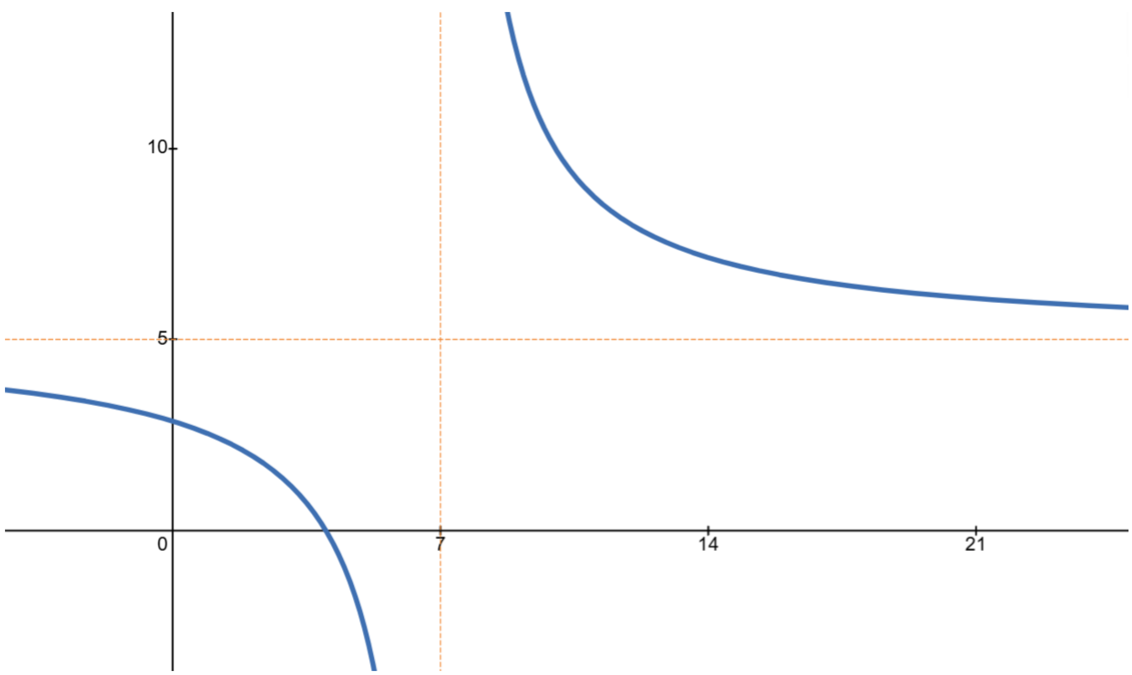
Asymptotes and Shape Example:

Consider

$$y = \frac{5x-20}{x-7} = \frac{5x+(-20)}{x+(-7)}.$$

Here $a = 5$, $b = -20$, $c = -7$.

- Vertical asymptote: $x = -c = 7$, Horizontal asymptote: $y = a = 5$
- y-intercept at $\left(0, \frac{20}{7}\right)$, x-intercept at $\left(\frac{20}{5}, 0\right) = (4, 0)$



Graph of $y = \dfrac{5x-20}{x-7}$ with asymptotes $x = 7$ and $y = 5$.

Example 1 (Three data points):

$$f(0) = 0, \quad f(10) = 5, \quad f(20) = 8, \qquad f(x) = \frac{ax+b}{x+c}.$$

Step 1. Use $f(0) = 0$: $f(0) = \dfrac{b}{c} = 0 \implies b = 0$. So $f(x) = \frac{ax}{x+c}$.

Step 2. Use $f(10) = 5$: $\dfrac{10a}{10+c} = 5 \implies 10a = 50 + 5c$. (1)

Step 3. Use $f(20) = 8$: $\dfrac{20a}{20+c} = 8 \implies 20a = 160 + 8c$. (2)

Step 4. Solve (1)–(2): From (1), $c = 2a - 10$. Substitute into (2):

$$20a = 160 + 8(2a - 10) = 160 + 16a - 80 = 16a + 80 \Rightarrow 4a = 80 \Rightarrow a = 20.$$

Then $c = 30$. Final model:

$$f(x) = \frac{20x}{x+30}.$$

Check: $f(10) = 5$, $f(20) = 8$, $f(0) = 0$ — all match.

Example 2 (Two data points + horizontal asymptote):

$$f(10) = 16, \quad f(30) = 18, \quad \text{horizontal asymptote } y = 20.$$

Model: $f(x) = \frac{ax+b}{x+c}$.

Step 1. Use the asymptote: $y = a = 20 \Rightarrow f(x) = \frac{20x+b}{x+c}$.

Step 2. Use $f(10) = 16$: $\dfrac{200+b}{10+c} = 16 \Rightarrow b = 16c - 40$.

Step 3. Use $f(30) = 18$: $\dfrac{600+b}{30+c} = 18 \Rightarrow b = 18c - 60$.

Step 4. Substitute: $16c - 40 = 18c - 60 \Rightarrow 2c = 20 \Rightarrow c = 10$. Then $b = 16(10) - 40 = 120$.

Final model:

$$f(x) = \frac{20x + 120}{x + 10}.$$

Check:

$$f(10) = \frac{320}{20} = 16, \quad f(30) = \frac{720}{40} = 18,$$

and the horizontal asymptote $y = 20$ matches perfectly.

Summary for Chapter 14:

- Identify asymptotes directly from $\dfrac{ax+b}{x+c}$.
- Recognize the hyperbola shape around those lines.
- Build models by solving for a, b, c using two or three conditions.