Chapters 10–11: Exponential Models

This document provides a concise review of exponential models.

Key Concepts

- General Exponential Model: $y(t) = A\,b^t$.
- Two exponential formulas used by banks:
  - Discrete compounding: $B(t) = A\,\left(1 + \frac{r}{n}\right)^{nt}$.
  - Continuous compounding: $B(t) = A\,e^{rt}$.

Example 1: Model from Two Data Points

Assume t is years after 1960. The U.S. minimum wage was \$1.60 in 1960 and \$2.30 in 1968. Find $w(t)$ and compute the value it predicts for the minimum wage in the year 2025.

Solution: Let $w(t) = Ab^t$. From $w(0) = Ab^0 = 1.60$ and $w(8) = Ab^8 = 2.30$ we get $A = 1.60$ and $b = \left(\frac{2.30}{1.60}\right)^{1/8} \approx 1.04640783$. Thus $w(t) = 1.60 \cdot (1.04640783)^t$.
Prediction for 2025 ($t = 65$): $w(65) = 1.60 \cdot (1.04640783)^{65} \approx \$\,30.53$.

Example 2: Doubling Time with One Data Point

A population of a town doubles every 6 years, and the population in 10 years will be 8000, so $P(10) = 8000$. Find $P(t)$.

Solution: Doubling every 6 years means this population can be *modeled* with an exponential function. Let $P(t) = Ab^t$. From $P(6) = Ab^6 = 2A$ and $P(10) = Ab^{10} = 8000$, we get $b = 2^{1/6} \approx 1.12246205$ and $A = \frac{8000}{b^{10}} \approx \frac{8000}{(1.12246205)^{10}} \approx 2519.84$.
Therefore we have $P(t) = 2519.84 \cdot (1.12246205)^t$.

Example 3: Half-Life to Discrete Factor

A substance has half-life 5 days, and there are 40 grams today. Write a model for the mass $m(t)$ after t days.

Solution: Half-life every 5 days means the mass of this substance can be *modeled* with an exponential function. Let $m(t) = Ab^t$. From $m(5) = Ab^5 = \frac{1}{2}A$ and $m(0) = Ab^0 = 40$, we get $\Rightarrow b = 2^{-1/5} \approx 0.87055056$ and $A = 40$.
Thus, $m(t) = 40 \cdot (0.87055056)^t$.

Example 4: Continuous Compounding

One account earns interest at 7% annually compounded monthly and another account earns interest at 7% annually compounded continuously. If both accounts start with \$2,000, what is the balance of each account in 30 years?

Solution:

- Discrete compounding: $B_1(30) = 2000\,\left(1 + \frac{0.07}{12}\right)^{12\cdot 30} \approx \$\,16232.99$.
- Continuous compounding: $B_2(30) = 2000\,e^{0.07\cdot 30} \approx \$\,16332.34$.