

Ch 11: Modeling Recap plus $y = e^x$

Goal: Learn the number e . Review the concept of modeling.

Entry Task: (Conroy Fall 2011 Exam 2)

$y = a e^{rx}$ is a special case of our exponential model
 a is still the y-intercept and $b = e^r$
 $r > 0$ gives exponential growth
 $r < 0$ gives exponential decay

3. City A’s population doubles every 19 years. City B’s population triples every 28 years.
In the year 2000, City A’s population was 40,000.
In 2010, City B’s population was twice that of City A.
When will City B’s population reach 500,000? Express your answer in years after 2000.

Read and plug in the given info to the models, then stop...

• **City A:** $y = a \cdot b^t$

• **City B:** $y = a \cdot b^t$

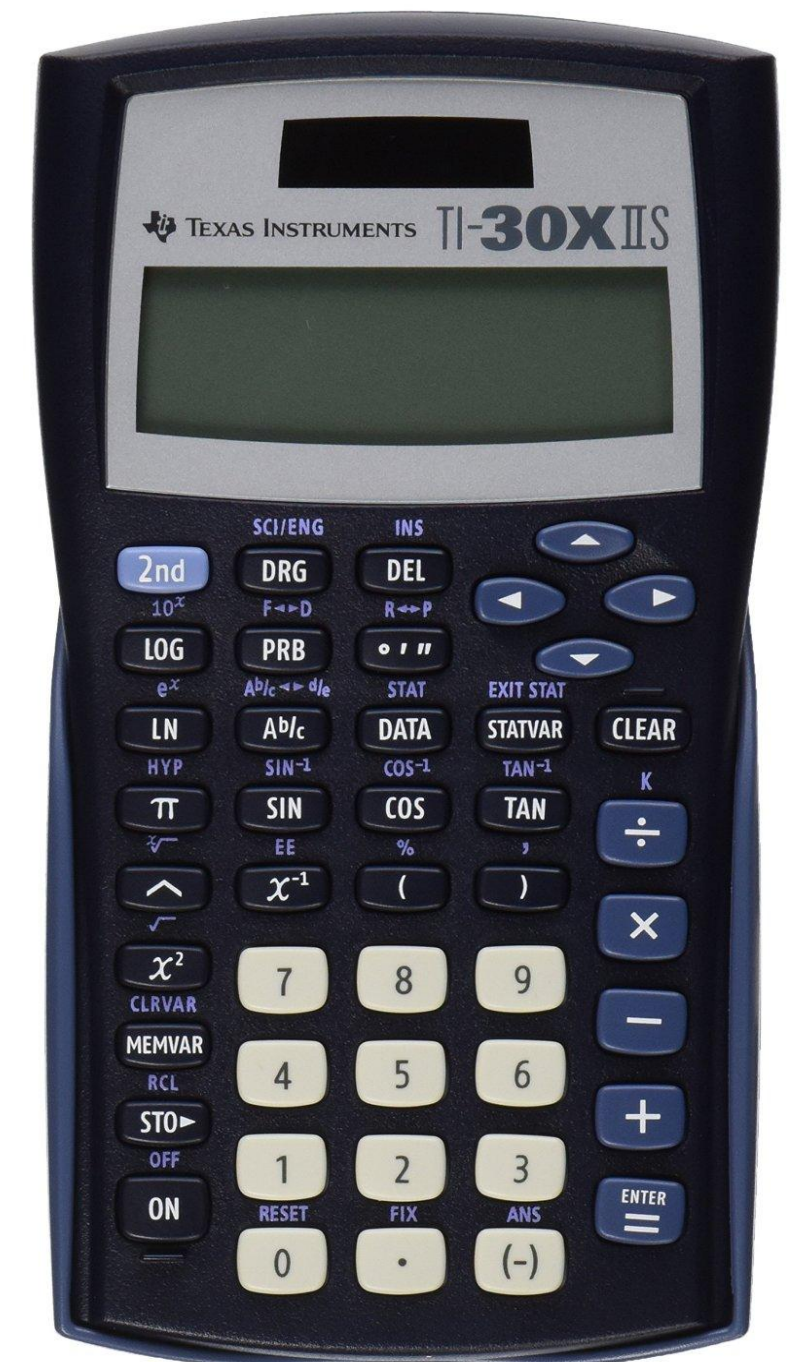
Some Graphs and Examples (*get your calculator out*)

$$y = a e^{rx}$$

Note: a is still the y-intercept and $b = e^r$

Example 1: Let $a = 5$ and $r = 0.03$, then what does the graph look like?

Example 2: If $a = 100$ and $r = -0.05$, then what does the graph look like?



A peek at Ch 11 HW, can you do this now?

In 1967, the U.S. minimum wage was \$1.40 per hour. In 1979, the minimum wage was \$2.90 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where t represents the time in years after 1960.

(a) Find a formula for $w(t)$. (Round values to three decimal places.)

$w(t) =$

(b) What does the model predict for the minimum wage in 1960? (Round your answer to the nearest cent.)

\$

How about this one?

A certain town is experiencing a population boom. In 1990, the population was 850 and five years later it was 1240.

(a) Find a linear model $I(x)$ for the population of the town in the year $1990 + x$.

$I(x) =$

Find an exponential model $p(x)$ for the population of the town in the year $1990 + x$. (Round values to three decimal places.)

$p(x) =$

How about this one? (yes that is almost all of chapter 11!)

In 1989, research scientists published a model for predicting the cumulative number of AIDS cases reported in the United States:

$$a(t) = 155\left(\frac{t - 1980}{10}\right)^3, \quad (\text{thousands})$$

where t is the year. This paper was considered a "relief," since there was a fear the correct model would be of exponential type. Use the two data points predicted by the research model $a(t)$ for the years 1989 and 1999 to construct a *new* exponential model $b(t)$ for the number of cumulative AIDS cases. (Round values to three decimal places. Let t represent the years after 1980.)

$b(t) =$, (thousands)

The symbol “e” appears in many applications.

Here is one that gives you a chance to play around with e^x while introducing you to some special functions.

Example: Consider the two functions

$$f(t) = \frac{e^t + e^{-t}}{2} \text{ and } g(t) = \frac{e^t - e^{-t}}{2}$$

(a) Plot several points and try to graph these functions

(b) *Algebra challenge:* Expand and completely simplify
 $(f(t))^2 - (g(t))^2$

Note:

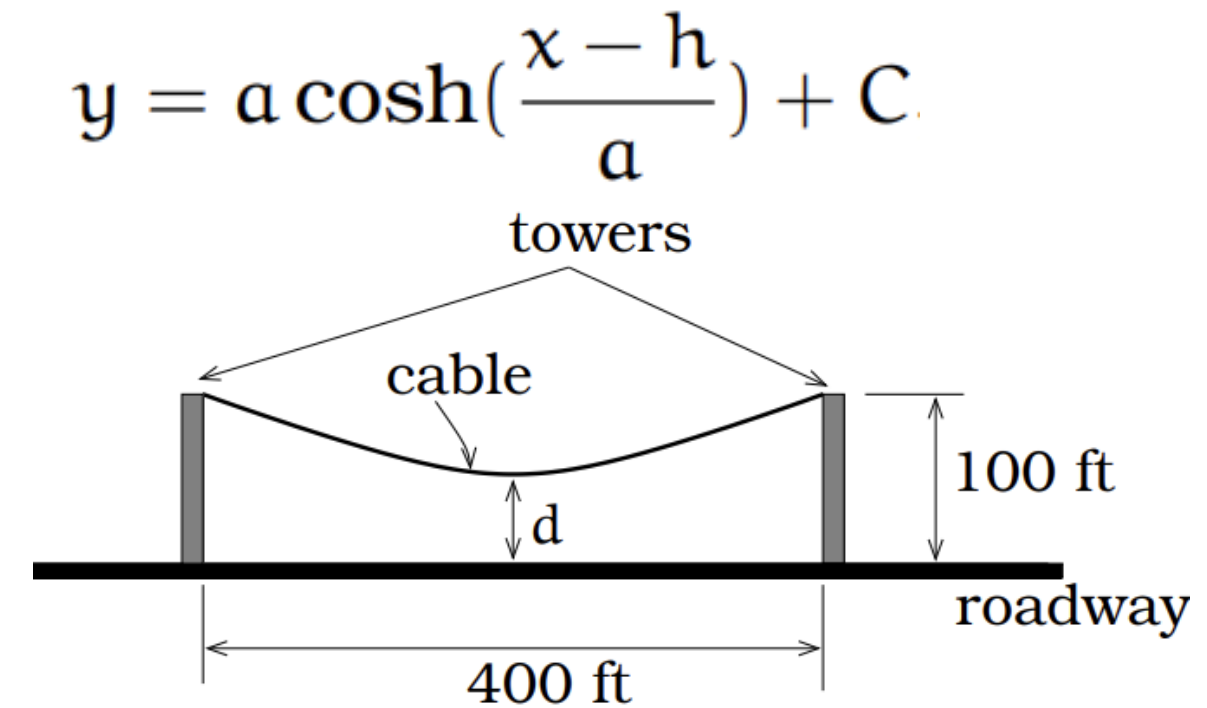
- $x^2 + y^2 = 1$ is a **circle**. (We know that one)
- $x^2 - y^2 = 1$ is called a **hyperbola**.

The functions $\frac{e^t + e^{-t}}{2}$ and $\frac{e^t - e^{-t}}{2}$ have funny names, they are called hyperbolic cosine and hyperbolic sine and are often given the function names:

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

I personally find $y = \cosh(x)$ to be a very interesting function. The curve it makes is called a catenary and you see them **every day**!

Impose a coordinate system so that the picture is symmetric about the y-axis and the roadway coincides with the x-axis. The hanging cable constant is $a = 500$ and $h = 0$. Use the model and the picture below



Find the minimum distance from the cable to the road.

Some background on e (do NOT need to know for exam)

Compounding Interest

We say interest is **compounding** if it is computed on the entire balance (principal and previous interest).

If P is the starting amount (*principal*) and r is the decimal annual rate, compounded n times a year, then a bank would use the formula:

$$F = P \left(1 + \frac{r}{n}\right)^{nt},$$

Example: You invest \$5000 at and annual rate of 3%

$$F = 5000 \left(1 + \frac{0.03}{n}\right)^{n t}$$

	n	Balance one yr.
semi-annual	2	\$5151.12500
quarterly	4	\$5151.69595
monthly	12	\$5152.07979
Daily	365	\$5152.26632
Hourly	87605	\$5152.27241
every minute	525600	\$5152.27267
every second	3153600	\$5152.27267

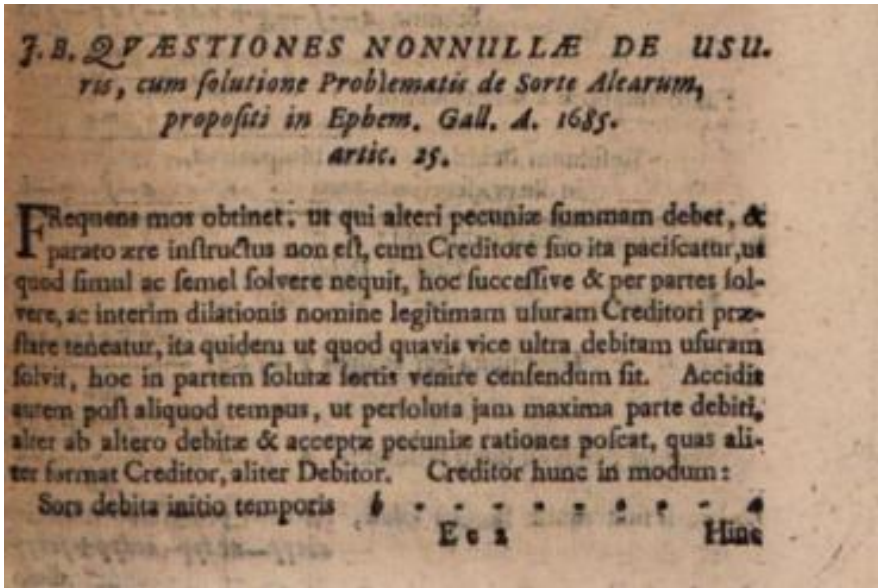
The value this is approaching what is called the value from **continuous compounding**. And it is also given by

$$F = Pe^{rt}$$
$$F = 5000e^{0.03(1)} \approx \$5152.27267$$

Some history, just for fun (again do NOT need for exam)



Figure 10. Jacob Bernoulli (1655-1705).



J. B. (Jacob Bernoulli). Some Questions on Interest, with the solution of a problem concerning the stakes of gamblers, proposed in the French Ephemerides, year 1685, article 25.

It often happens that someone who owes a sum of money to another, and is not equipped with ready cash, makes the following agreement with his creditor: since he cannot pay it all at once, he will pay it gradually and in installments; meanwhile, for the delay, the creditor retains only the lawful interest. Furthermore, if at any time he should pay more than what is owed in interest, that excess is to be counted as a portion of the principal repaid.

English Translation



I also observe, moreover, that this series is a geometric progression whose terms, increasing without bound, arise from the ratio $1 + \frac{1}{n}$; therefore, if a principal sum be divided into as many moments of time that in each moment the interest be one- n -th part of the principal, then the whole capital together with the accumulated interest will become $(1 + \frac{1}{n})^n$.

The greater the subdivision of the time, the smaller the interest of each instant, yet the greater the total product becomes; so that, if the number of moments be increased without bound, the gain from interest tends toward a fixed limit.

This number is about 2 and $\frac{3}{4}$, or a little greater; for brevity I shall denote it by the sign $\star$.

Thus it happens that, however frequently the interest be compounded, it never increases the capital in greater ratio than $\star$.

used because it was the first letter of his name. It may not even be the case that the e was derived from the first letter of the word “exponential”. At that time Euler had already assigned the notation a to another constant in his study and he used the letter e, which was the vowel after a to denote and represent Euler’s number e. The reason, whatever it may be, led to the allocation of e to the con-

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

In this piece of work, Euler stated an approximation for number e to 18 decimal places

$$e = 2.718281828459045235.$$

It remains a mystery of how he went about to derive e to 18 decimal places. He listed it without showing any calculations on how it was done. The most applicable explanation is taking about 20 terms of

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$



Figure 14. Leonhard Euler (1707-1783).