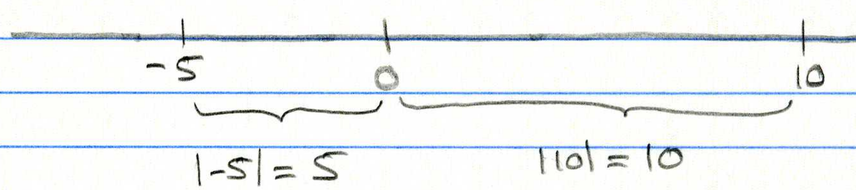
Dr. Loveless

The Absolute Value

The absolute value is a basic and valuable function in calculus that seems to occasionally confound students of calculus. The following pages give a quick discussion of the function and how it effects you.

The absolute value, $f(x) = |x|$, gives the "magnitude" of a number. Visually, $|x|$ gives the distance that x is away from 0 on the number line.

Ex)



For actual calculation, we see that the absolute value performs two distinct operations depending on the sign of the input.

① If x is positive (or zero), then $\boxed{\text{DO NOTHING}}$
That is, $|x| = x$, if $x \geq 0$.

Ex) $|10| = 10$ $\qquad |7| = 7$

$|2^{50}| = 2^{50}$ $\qquad |x^2| = x^2 \leftarrow$ { BECAUSE $x^2 \geq 0$ ALWA

$|e^x| = e^x \leftarrow$ { BECAUSE $e^x > 0$ ALWA

② If x is negative, then $\boxed{\text{FLIP THE SIGN}}$
That is, $|x| = -x$, if $x < 0$

Ex) $|-2| = -(-2) = 2$

$|-50| = -(-50) = 50$

$|-x^2| = -(-x^2) = x^2$.

Thus, $|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$

Since the absolute value has two distinct operations depending on the input, whenever you encounter an absolute value you MUST think in terms of two cases:

To work with $|BLAH|$, you must consider

① CASE 1: If $BLAH \geq 0$, then you can replace $|BLAH| = BLAH$.

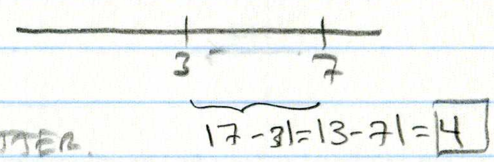
② CASE 2: If $BLAH < 0$, then you can replace $|BLAH| = -BLAH$.

APPLICATIONS

Distances The positive distance between two numbers, a and b, on the number line is given by $|a - b|$.

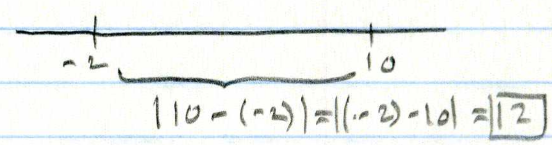
Ex

$|3-7| = 4$
$|7-3| = 4$



NOTE: ORDER DOESN'T MATTER.

$|10-(-2)| = 12$
$|(-2)-10| = 12$



Inequalities A quick way to say $-a \le x \le a$ is to write $|x| \le a$.

Ex) $|x| \le 6$ means that x is within 6 of 0, which is the same as saying $-6 \le x \le 6$. These are <u>identical</u> inequalities

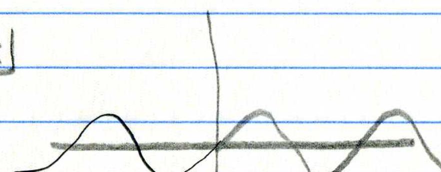
Ex) $|x - 20| < 3$ means that x is within 3 of 20, which is the same as

$-3 < x - 20 < 3$ these are all <u>identical</u> inequalities

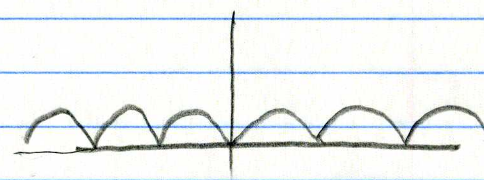
$17 < x < 23$

Making Functions Positive Since $|\text{BLAH}|$ = positive magnitude of BLAH, the absolute value of a function makes all outputs positive.

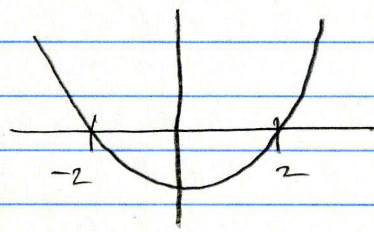
Ex)



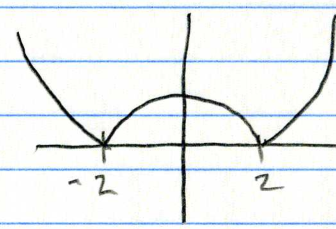
$\boxed{y = \sin(x)}$



$y = |\sin(x)|$



$y = x^2 - 4$



$y = |x^2 - 4|$

In essence, all positive y's stay positive and we get the mirror image of all negative y's.

Examples

- $\lim_{x \to 10} |x+1| + |x-12|$

when x is near 10, $x+1$ is POSITIVE

so $|x+1| = x+1$.

when x is near 10, $x-12$ is NEGATIVE

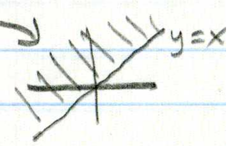
so $|x-12| = -(x-12)$

Hence, the functions become

$$\lim_{x \to 10} (x+1) + -(x-12) = (10+1) - (10-12) = \boxed{13}$$

> NOTE HOW YOU **MUST** CONSIDER WHAT IS INSIDE THE ABSOLUTE VALUE BEFORE YOU CAN ATTACK THE PROBLEM.

- Graph $|y-x| = 4$



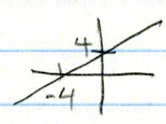
CASE 1 $\quad y - x \geq 0 \iff y \geq x$

Then $|y-x| = 4$ becomes

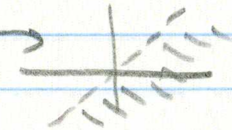
$y - x = 4$, so

$y = x+4$

Thus, one part of the graph will be



$y = x+4$ when $y \geq x$, which always happens on this line.



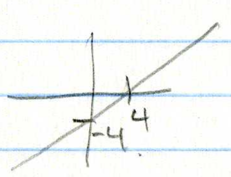
CASE 2 $\quad y - x < 0 \iff y < x$

Then $|y-x| = 4$ becomes

$-(y-x) = 4$, so

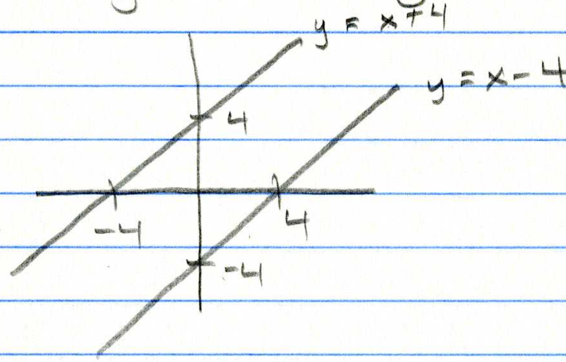
$-y + x = 4$

$y = x - 4$



when $y < x$ which always happens on this line.

So the graph of $|y-x|=4$ is



• Solve $|x^2-8|=17$

$\boxed{\text{CASE 1}}$ $x^2-8 \geq 0 \iff x^2 \geq 8$

Then $|x^2-8|=17$ becomes

$$x^2-8=17$$

$$x^2=25$$

$$x=\pm 5$$ and both satisfy the condition $x^2 \geq 8$ ✓

$\boxed{\text{CASE 2}}$ $x^2-8<0 \iff x^2<8$

Then $|x^2-8|=17$ becomes

$$-(x^2-8)=17$$

$$-x^2+8=17$$

$$-x^2=11$$

$$x^2=-11$$ NO REAL SOL'NS

Thus, the only sol'ns are $\boxed{x=\pm 5}$.

IN GENERAL, YOU **MUST** EXPLORE ABSOLUTE VALUES IN TWO CASES.

I HOPE THIS HELPS.