## 9.9 Overview

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections.

### 9.9: Derivative Applications

First observe the big connections between the 'original' function and the 'derived' function.

| ORIGINAL ($f(x)$) | DERIVATIVE ($f'(x)$) |
|---|---|
| $f(x)$ = height of original at $x$ | $f'(x)$ = slope of original at $x$ |
| increasing (uphill left-to-right) | positive (above $x$-axis) |
| decreasing (downhill left-to-right) | negative (below $x$-axis) |
| horizontal tangent | zero (crosses $x$-axis) |

These are the fundamental facts that we use over and over again in applications. Eventually you will be able to use these even if you don't have a graph in front of you. These connections should be something you natural use without thinking about by the end of the term. At this point you have the skills to do the following two tasks:

1. Given the original graph, draw a rough sketch of derived graph (you should be able to clearly draw where the derived graph crosses the $x$-axis, is below the $x$-axis, and is above the $x$-axis).

2. Given the derived graph, draw a very rough sketch of the general shape of the original graph (you should be able to clearly draw on the original graph the $x$-coordinates of the tops of 'hill' and bottoms of 'valleys' along with where the graph should be increasing and where it should be decreasing).

Now let's discuss some business specific examples (for a more detailed reviewed of the definitions of business terms see my business terms that I posted last week).

1. A **demand function** gives a relationship between price, $p$, and the number of units that will sell at that price (this is the demand). We typically use $q$ or $x$ for the demand quantity.

   - Often when we are just discussing demand in business, we are given $q$ (or $x$) in terms of $p$. For example, you often see things like $q = 20 - 2p$ which gives the demand in terms of price. But if we want to study revenue, then we typically want all variables in terms of quantity. In which case we could rearrange the equation to give $p$ in terms of $q$. That just requires some basic algebra, in the case of $q = 20 - 2p$, it would become $p = \frac{20-q}{2} = 10 - \frac{1}{2}q$ (which the book would write at $p = 10 - \frac{1}{2}x$).
   - One typical economic/business property of a demand function is: If the price goes up, the demand goes down (and vice versa). In my example, $q = 20 - 2p$ you see that this would be a decreasing line (it wouldn't make sense for it to be an increasing line).
   - *Aside*: The demand function is something typically dictated by the consumer and the open market (we would get it from historical information or a market analysis). We often don't have control over it, but we might have some control over the price we sell at and the quantity we produce and sell (I say 'might', because in a real situation we would have to worry about competition which might force us into a certain price point). In this class, we focus on idealized situations where we have complete control over the quantity and price. Things are rarely this simple, but it is an important starting point in any analysis. And you can use the same methods of this class to analysis and optimize profit even if there are more restrictions on price and quantity (you would just adjust the analysis for those restrictions).

- Recall that
$$TR(q) = \text{PRICE} \cdot \text{QUANTITY} = p \cdot q.$$
So if you are given a demand function for $p$ in terms of $q$, then you just multiply by $q$ to get the revenue function. In the example $p = 10 - \frac{1}{2}q$, you get $TR(q) = (10 - \frac{1}{2}q)q = 10q - \frac{1}{2}q^2$.
- In some situations price is a fixed constant. For example, if the price was fixed at \$30 per item, then the revenue is just $TR(q) = 30q$. (This is a situation where price is dictated by competition or for some other reason, so we aren't doing a demand analysis).

2. The average cost per item gives the overall average cost for producing $q$ items.

   - Sometimes this information is easier to work with or more readily available then the total cost (for example, you may have some information about average cost from historical information or from standard models in your industry).
   - Recall that $AC(q) = \frac{TC(q)}{q}$. So
   $$TC(q) = \text{QUANTITY} \cdot \text{AVERAGE COST} = qAC(q).$$
   - There are a couple of problems in the homework of 9.9 that involve average cost (9.9.035 and 9.9.037, which is currently 5 and 6 of the 9.9 homework).

   Now to the calculus:

3. Recall that
$$MR(q) = TR'(q) \text{ and } MC(q) = TC'(q).$$
Marginal Revenue and Marginal Cost (as defined here) give the instantaneous rate of change of Revenue and Cost. We interpret these as the approximate change in revenue and the change in cost if one more item is sold or produced.
$$\text{Profit} = P(q) = TR(q) - TC(q) \text{ and Marginal Profit} = MP(q) = P'(q) = MR(q) - MC(q).$$
Marginal Profit (as defined here) gives the instantaneous rate of change of Profit. Again, we interpret it as the approximate change in Profit if one more unit is sold.

   **WARNING**: Capital 'P' is profit, and lower case 'p' is price. We unfortunately have two things for which we like to use the letter 'p', so you need to be carefully about whether you are using upper case 'P' for profit or lower case 'p' for price. You should be able to always tell from context as well, and you can always ask a tutor (or ask your TA during class for clarification). Just make sure you are clear in your own work, so we can understand what you are doing.

4. Using the fundamental observations from the beginning of this review sheet you can answer many questions about $TR$, $TC$ and $P$ by using their derivative $MR$, $MC$, and $MP$. Here are some questions:

   - *Question*: Where does $TR$ have a horizontal tangent?
   *Answer*: Solve when $MR(q) = 0$.
   - *Question*: Give the intervals where $TR$ is increasing.
   *Answer*: First find where $MR(q) = 0$, then think about the values of $MR(q)$ before and after the quantities you just found (anywhere $MR$ is positive, remember that $TR$ is increasing).

- *Question*: Find where profit is decreasing.
  *Answer*: First find where $MP(q) = 0$, then think about the values of $MP(q)$ before and after the quantities you just found (anywhere $MP$ is negative, remember that Profit is decreasing).
- *Question*: Find where profit is maximized.
  *Answer*: Find where $MP(q) = 0$ (or equivalently, find when $MR(q) = MC(q)$). If there are multiple values, you are going to need to think about the values of $MP$ before and after the quantities you just found. You are looking for the location on the profit where it switching from increasing, to a high point, to decreasing (so you are looking for places where $MP$ changes from positive, to zero, to negative).
- And so on and so on ...
  You are just using these same connections over and over again. Once you understand the connections between derivatives and the original function, these questions should be easy to answer.

  It's typically a good starting point to find the derivative of whatever you are trying to study. Then solve for when the derivative is equal to zero. Then make appropriate conclusions.

The same ideas apply to other types of functions including distance/speed analysis and studying functions in general. Most of the questions are exactly like those above. But let's consider an example where we are comparing two speed functions (often a point of confusion for students).
*Example*: Assume we have two balloons. Their heights are given by $A(x)$ and $B(x)$ in feet. Here are some general observations about comparing differences between the heights of the two balloons:

1. Suppose $A(x)$ is **bigger** than $B(x)$ (so balloon $A$ is higher $B$) at a particular time.
   (a) If the rate of ascent (speed) $A'(x)$ is bigger than $B'(x)$, then the distance between them is growing at this particular time (A is above B and A is going faster upward then B in this scenario). Written concisely, if $A(x_0) > B(x_0)$ and $A'(x_0) > B'(x_0)$, then the difference between $A(x)$ and $B(x)$ is increasing.
   (b) If the rate of ascent (speed) $A'(x)$ is smaller than $B'(x)$, then the distance between them is shrinking at this particular time (A is above B and B is going faster upward than A in this scenario). Written concisely, if $A(x_0) > B(x_0)$ and $A'(x_0) < B'(x_0)$, then the difference between $A(x)$ and $B(x)$ is decreasing.

With these observations and the general facts about derivatives, you have all the tools to answer questions about increasing, decreasing, maximum points, minimum points, farthest apart, closest together, etc. Now go practice on the 9.9 homework.