13.2 Overview

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections.

13.4: Definite Integrals

We now know how to find antiderivatives and solve for the constant of integration. We now connect the graphical concept of areas with antiderivatives in what is called the fundamental theorem of calculus. There are two mains facts in this section:

1. We defined $\int_a^b f(x)\,dx$ to be the change in the antiderivative from a to b. In other words, the fundamental theorem of calculus says if $A(x)$ is *any* antiderivative of $f(x)$ (meaning $A'(x) = f(x)$), then
$$\int_a^b f(x)\,dx = A(b) - A(a).$$

2. We observed that $\int_a^b f(x)\,dx$ can be visualized as the signed area between $f(x)$ and the x-axis from a to b (meaning area above the x-axis counts as positive and area below the x-axis counts as negative).

Here is a quick summary of our two types of integrals:

Type:	Indefinite Integral	Definite Integral
Notation:	$\int f(x)\,dx$	$\int_a^b f(x)\,dx$
Represents:	general antiderivative	change in antiderivative from a to b
Output:	a function with "$+C$"	a number that visually can be seen as an area
Example:	$\int 3x^2\,dx = x^3 + C$	$\int_1^2 3x^2\,dx = x^3\Big\|_1^2 = (2)^3 - (1)^3 = 7$

On the next page, I give examples where you just practice mechanically computing definite integrals. On the page after that I go through a visual example where you think about areas.
Here is how to compute a definite integral:

1. Find the general antiderivative (you can put in the +C or leave it off, it doesn't matter for definite integrals as we discuss in class).

2. Plug in the bounds and subtract! That's it.

Examples of computing definite integrals Compute the following numbers.

- $\displaystyle\int_1^3 2x+1\,dx$

  ANSWER:
  $\displaystyle\int_1^3 2x+1\,dx = x^2+x\Big|_1^3 = [(3)^2+(3)]-[(1)^2+(1)] = 12-2 = 10.$

- $\displaystyle\int_1^4 \frac{3}{\sqrt{x}}+2\,dx$

  ANSWER:
  $\displaystyle\int_1^4 3x^{-1/2}+2\,dx = 6x^{1/2}+2x\Big|_1^4 = [6(4)^{1/2}+2(4)]-[6(1)^{1/2}+2(1)] = 20-8 = 12.$

- $\displaystyle\int_1^e \frac{5}{x}\,dx$

  ANSWER:
  $\displaystyle\int_1^e 5\frac{1}{x}\,dx = 5\ln(x)|_1^e = [5\ln(e)]-[5\ln(1)] = 5-0 = 5.$

- $\displaystyle\int_0^2 e^{3x}\,dx$

  ANSWER:
  $\displaystyle\int_0^2 e^{3x}\,dx = \frac{1}{3}e^{3x}\Big|_0^2 = \left[\frac{1}{3}e^{3(2)}\right]-\left[\frac{1}{3}e^{3(0)}\right] = \frac{1}{3}e^6-\frac{1}{3}.$

- $\displaystyle\int_0^1 x^2(4-x)\,dx$

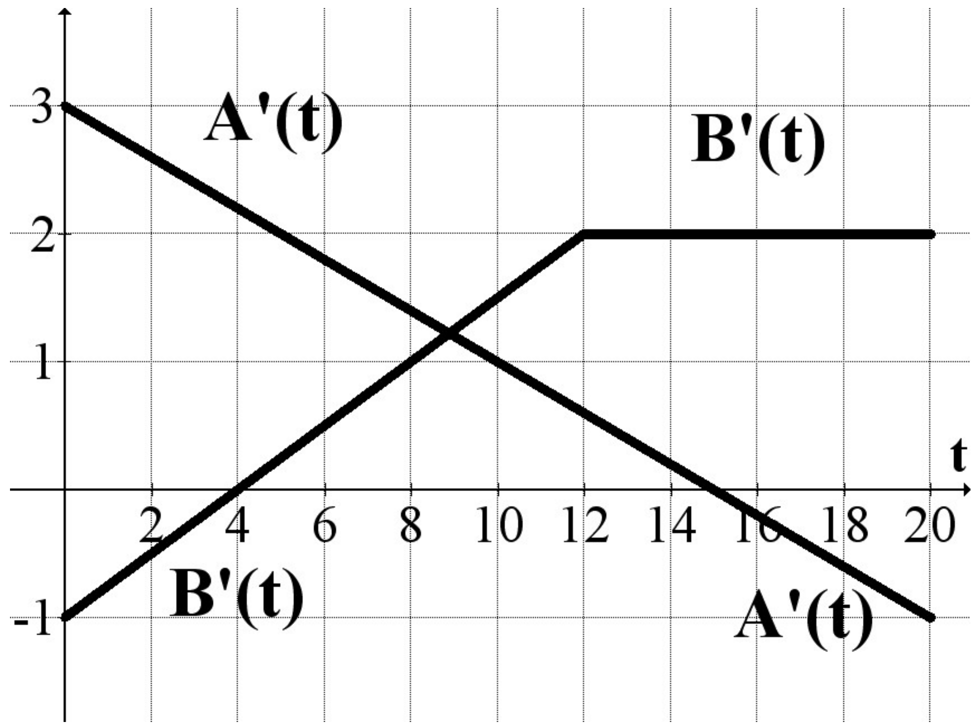
  ANSWER:
  $\displaystyle\int_0^1 4x^2-x^3\,dx = \frac{4}{3}x^3-\frac{1}{4}x^4\Big|_0^1 = \left[\frac{4}{3}(1)^3-\frac{1}{4}(1)^4\right]-\left[\frac{4}{3}(0)^3-\frac{1}{4}(0)^4\right] = \left[\frac{4}{3}-\frac{1}{4}\right]-0 = \frac{13}{12}.$

- $\displaystyle\int_1^2 \frac{x^2+x}{x^4}\,dx$

  ANSWER:
  $\displaystyle\int_1^2 x^{-2}+x^{-3}\,dx = -x^{-1}-\tfrac{1}{2}x^{-2}\Big|_1^2 = \left[-\tfrac{1}{(2)^1}-\tfrac{1}{2(2)^2}\right]-\left[-\tfrac{1}{(1)^1}-\tfrac{1}{2(1)^2}\right] = \left[-\tfrac{1}{2}-\tfrac{1}{8}\right]-\left[-1-\tfrac{1}{2}\right] = \tfrac{7}{8}.$

Now for some interpretation. We visually saw in class that area under $f(x)$ from $x = a$ to $x = b$ is the same as $\int_a^b f(x)\,dx$. This gives us a visual way to think about antiderivatives when given a rate or derivative graph. Here is an example from an old exam:

Question: The rate of ascent graphs are given for two balloons. In other words, $A(t)$ and $B(t)$ give the height of the balloons and $A'(t)$ and $B'(t)$ give the rate of ascent of the balloons. At time $t = 0$ both balloons start at 40 feet (so $A(0) = 40$ and $B(0) = 40$). All times are in minutes and all heights are in feet.



1. (3 pts) Approximate $\int_6^{14} A'(t)dt$.

   ANSWER: $\int_6^{14} A'(t)dt =$ ________________

2. (3 pts) Give the height of Balloon B at $t = 2$ seconds.

   ANSWER: $B(2) =$ ________________ feet

3. (3 pts) Find a time t, other than zero, then $B(t) = B(0)$.

   ANSWER: $t =$ ________________

4. (4 pts) Give the maximum height of balloon A on the interval $t = 0$ to $t = 24$.

   ANSWER: max height $=$ ________________ feet

5. (4 pts) Find the distance between the balloons when balloon B is farthest above balloon A.

   ANSWER: ________________ feet

Try these on your own first, answers on the next page.

Some of these answers are approximate, so you won't have to get exactly what I have, but you would need to be very close:

1. $\int_6^{14} A'(t)dt =$ the signed area between $A'(t)$ and the t-axis from 6 to 14. This is a trapezoid. Area $= \frac{1}{2}(1.75 + 0.25)8 = 8$ feet

2. To find the height of balloon B at time 2, we need to find the change in height from 0 to 2, then use the initial information.
Change in height $= B(2) - B(0) = \int_0^2 B'(t)\,dt = -\frac{1}{2}(1 + 0.5)2 = -1.5$ feet. Since it started at 40 feet and went down 1.5 feet, the height is $B(2) = 38.5$ feet.

3. We are trying to find when the height of balloon B will get back up to 40 again. That will only happen when the change in height is zero. Meaning we need to find a location when the signed area cancels out and gives no net change. Since the area from 0 to 4 will be negative and since it is a straight line here, the area from 4 to 8 will exactly cancel off this area. In other words $B(8) - B(0) = 0$. Thus, at $t = 8$ the balloon will be at height 40 feet again.

4. We should know well that the maximum of $A(t)$ occurs when $A'(t)$ changes from positive to negative, which would be at $t = 15$. Now we just need to compute $A(15)$. We need to find the change in height from 0 to 15, then use the initial information.
Change in height $= A(15) - A(0) = \int_0^{15} A'(t)\,dt = -\frac{1}{2}(3)(15) = 22.5$ feet. Since it started at 40 feet and went up 22.5 feet, the height is $A(15) = 62.5$ feet.

5. We should be able to read quickly that A initially goes above B in height. Then at $t = 9$, A is farthest above B. After that B catches up and eventually passes A. So B is going to be farthest above A at the very end, $t = 20$. Now let's compute some areas:
Change in height of $B = B(20) - B(0) = \int_0^{20} B'(t)\,dt = -\frac{1}{2}(1)(5) + \frac{1}{2}(2)(8) + (2)(8) = -2.5 + 8 + 16 = 21.5$ feet. Since it started at 40 feet and went up 21.5 feet, the height is $B(20) = 61.5$ feet.
Change in height of $A = A(20) - A(0) = \int_0^{20} A'(t)\,dt = 22.5 - \frac{1}{2}(1)(5) = 22.5 - 2.5 = 20$ feet. Since it started at 40 feet and went up 20 feet, the height is $A(20) = 60$ feet.
For a difference of $B(20) - A(20) = 61.5 - 60 = 1.5$ feet (that is the farthest B ever gets ahead of A).