10.1, 10.2, and 10.3 Overview

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems, worksheets and lecture material. Hopefully this review sheet will remind you of some of the key ideas of these sections.

Terminology: Given a function $y = f(x)$.

- *Critical Number*: A *critical number* (or *critical value*) is a number $x = a$ at which $f'(a) = 0$. (*i.e.* the critical numbers are all the x values when the original graph has a horizontal tangent).
- *Critical Point*: A *critical point* is the (x, y) coordinates that correspond to a critical number. (Meaning you plug in your critical number back into the original function and find the corresponding y as well).
- *Relative Maximum*: A *relative maximum* (or *local maximum*) is a critical point at which the function changes from the increasing to decreasing.
- *Relative Minimum*: A *relative minimum* (or *local minimum*) is a critical point at which the function changes from the decreasing to increasing.
- *Horizontal Point of Inflection*: A *horizontal point of inflection* is a critical point at which the function does not change direction (*i.e.* increasing to increasing or decreasing to decreasing).
- *Concave Up*: At $x = a$, a function is *concave up* if the graph of the function is above the tangent line drawn at $x = a$. (*i.e.* the tangent slopes are increasing).
- *Concave Down*: At $x = a$, a function is *concave down* if the graph of the function is below the tangent line drawn at $x = a$. (*i.e.* the tangent slopes are decreasing).
- *Point of Inflection*: At $x = a$, a function has a *point of inflection* if the concavity changes at $x = a$.

Summary of Basic Connections

ORIGINAL ($f(x)$)	DERIVATIVE ($f'(x)$)	SECOND DERIVATIVE ($f''(x)$)
$f(x)$ = height of original at x	$f'(x)$ = slope of original at x	
increasing (uphill left-to-right)	positive (above x-axis)	
decreasing (downhill left-to-right)	negative (below x-axis)	
horizontal tangent	zero (crosses x-axis)	
concave up		positive
concave down		negative
possible inflection point		zero

Analyzing a Function

At the end of a calculus course, every student should be able to quickly analyze any function and get the basic shape and features. The basic methods to study a function $y = f(x)$ are summarized below. The goals are to find and classify all maximum, minimum, and inflection points and to determine where the function is increasing, decreasing, concave up, and concave down. Here are the methods we've discussed (the first two are ALL you need, the other two 'tests' are just special summaries):

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1. *First Derivative Analysis*:
   (a) Critical Numbers: Find $f'(x)$ and solve $f'(x) = 0$.
   (b) Number Line: Draw a number line with tick marks for the critical values:
      i. Positive or Negative: Between each critical value, plug in a number to $f'(x)$ and determine if it is positive or negative.
      ii. Increasing or Decreasing: Indicate appropriately where the original function is increasing or decreasing.
   (c) Conclusions: Make appropriate conclusions about max/min and horizontal points of inflections.
2. *Second Derivative Analysis*:
   (a) Possible Points of Inflection: Find $f''(x)$ and solve $f''(x) = 0$.
   (b) Number Line: Draw a number line with tick marks for the possible points of inflection:
      i. Positive or Negative: Between each possible point of inflection, plug in a number to $f''(x)$ and determine if it is positive or negative.
      ii. Concave Up or Concave Down: Indicate appropriately where the original function is concave up or concave down.
   (c) Conclusions: Make appropriate conclusions about max/min and inflection points.

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3. *The First Derivative Test to Classify Critical Numbers:* The theorem commonly called the 'first derivative test' just states what we already know. Namely,
   - If $f'(a) = 0$ and $f'(x)$ changes from positive to negative, then a local maximum occurs at $x = a$.
   - If $f'(a) = 0$ and $f'(x)$ changes from negative to positive, then a local minimum occurs at $x = a$.
   - If $f'(a) = 0$ and $f'(x)$ does not change (positive to positive or negative to negative), then a horizontal point of inflection occurs at $x = a$.
4. *The Second Derivative Test to Classify Critical Numbers:* The theorem commonly called the 'second derivative test' again just states what we already know. Namely,
   - If $f'(a) = 0$ and $f''(x)$ is negative, then a local maximum occurs at $x = a$.
   - If $f'(a) = 0$ and $f''(x)$ is positive, then a local minimum occurs at $x = a$.
   - If $f'(a) = 0$ and $f''(x)$ is zero, then the test 'fails' (go use the first derivative test if this happens).

Examples:
Here are several examples:

1. Analyze the function $f(x) = x^4 - 2x^3$.

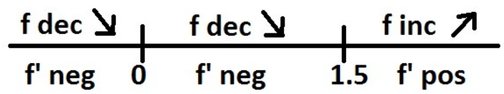
   First Derivative Facts:

   (a) *Critical Points*: First, we find $f'(x) = 4x^3 - 6x^2$. Solving gives

   $$\begin{aligned} 4x^3 - 6x^2 &= 0, && \text{factoring out } x^2 \text{ gives} \\ x^2(4x-6) &= 0 && \text{simplifying gives} \\ x &= 0 \text{ or } 6/4 \end{aligned}$$

   We get two critical numbers $x = 0$ and $x = \frac{6}{4} = \frac{3}{2} = 1.5$.

   (b) *Number Line*: We plug in -1, 1, and 2 (or any values before 0, between 0 and 1.5, and after 1.5) to the derivative and find that $f'(-1) = -10$ is negative, $f'(1) = -2$ is negative, and $f'(2) = 8$ is positive. We summarize:



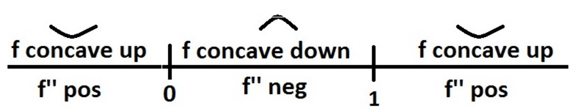
   Second Derivative Facts:

   (a) *Possible Points of Inflection*: First, we find $f''(x) = 12x^2 - 12x$. Solving gives:

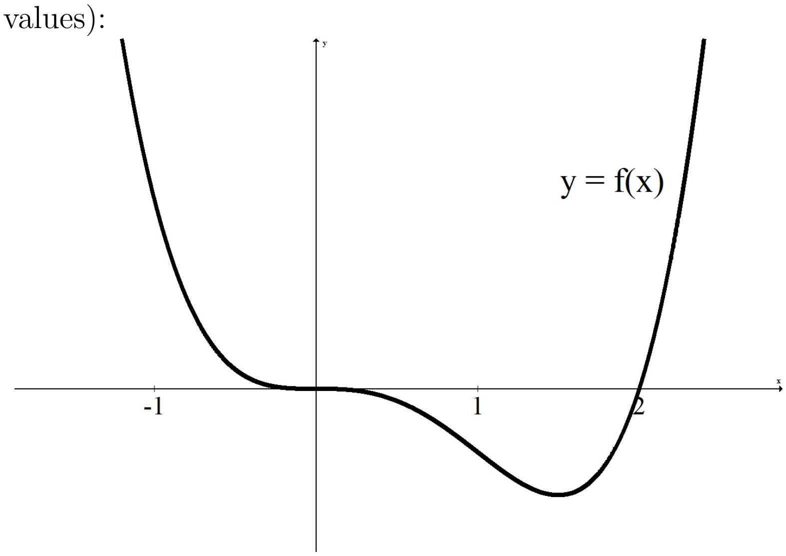
   $$\begin{aligned} 12x^2 - 12x &= 0, && \text{factoring out } 12x \text{ gives} \\ 12x(x-1) &= 0 \end{aligned}$$

   We get two possible points of inflection $x = 0$ and $x = 1$.

   (b) *Number Line*: We plug in -1, 1/2 and 2 (or any values before 0, between 0 and 1, and after 1) to the second derivative and find that $f''(-1) = 24$ is positive, $f''(1/2) = -3$ is negative and $f''(2) = 24$ is positive. We summarize:



   Summary: $x = 1.5$ gives a local minimum, $x = 0$ and $x = 1$ give points of inflection (Note $x = 0$ is a horizontal point of inflection). And we should be able to roughly sketch the shape of the function (for a more accurate sketch, plug these values back into the original function to get the corresponding y values):

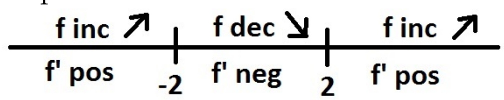


2. Analyze the function $f(x) = x^3 - 12x$.

First Derivative Facts:

(a) *Critical Points*: First, we find $f'(x) = 3x^2 - 12$. Solving $3x^2 - 12 = 0$ gives $x^2 = 4$. Thus, $x = -2$ and $x = 2$ are the critical numbers.

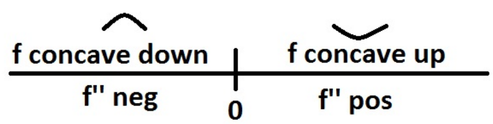
(b) *Number Line*: We plug in -3, 0, and 3 to the derivative and find that $f'(-3)$ is positive, $f'(0)$ is negative, and $f'(3)$ is positive. We summarize:



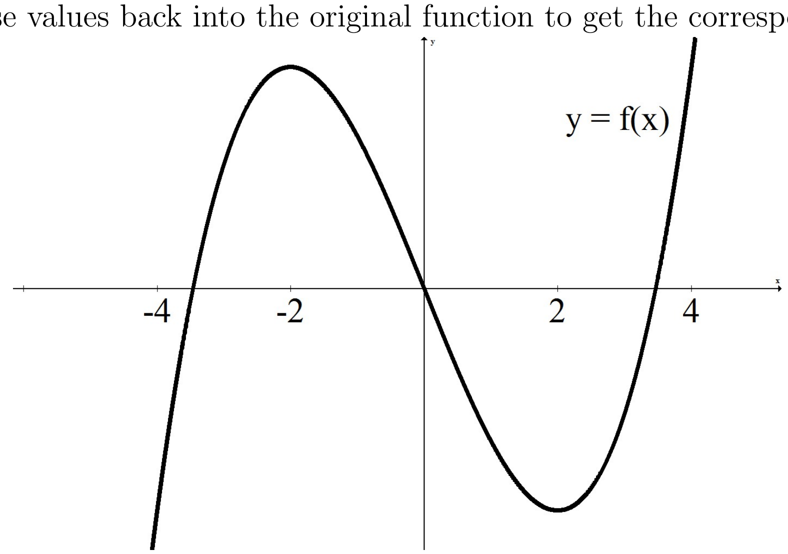
Second Derivative Facts:

(a) *Possible Points of Inflection*: First, we find $f''(x) = 6x$. Solving $6x = 0$ gives $x = 0$. Thus, $x = 0$ is the only possible point of inflection.

(b) *Number Line*: We plug in -1 and 1 to the second derivative and find that $f''(-1)$ is negative and $f''(1)$ is positive. We summarize:



Summary: $x = -2$ gives a local maximum, $x = 2$ gives a local minimum and $x = 0$ gives a point of inflection. And we should be able to roughly sketch the shape of the function (for a more accurate sketch, plug these values back into the original function to get the corresponding y values):



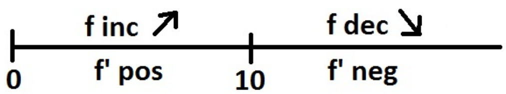
3. Analyze the function $f(x) = \frac{1}{x} - \frac{5}{x^2}$ for values $x > 0$. (Aside: there is a vertical asymptote at $x = 0$, so I just want you to focus on $x > 0$).

First Derivative Facts:

(a) *Critical Points*: First, since $f(x) = x^{-1} - 5x^{-2}$, we have $f'(x) = -x^{-2} + 10x^{-3}$. We simplify to get $f'(x) = -\frac{1}{x^2} + \frac{10}{x^3}$ and try to solve:

$$\begin{aligned} -\tfrac{1}{x^2} + \tfrac{10}{x^3} &= 0, && \text{multiplying by } x^3 \text{ gives} \\ -x + 10 &= 0 && \text{simplifying gives} \\ x &= 10 \end{aligned}$$

(b) *Number Line*: We plug in 1 and 15 (anything between 0 and 10 and anything after 10) to the derivative and find that $f'(1) = -1 + 10 = 9$ is positive and $f'(15) \approx -0.00148$ is negative. We summarize:

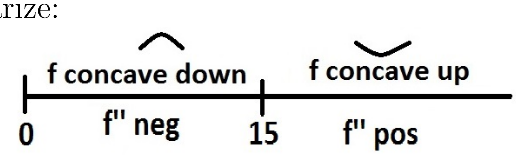


Second Derivative Facts:

(a) *Possible Points of Inflection*: First, we find $f''(x) = 2x^{-3} - 30x^{-4}$. We simplify to get $f''(x) = \frac{2}{x^3} - \frac{30}{x^4}$ and try to solve:

$$\begin{aligned} \tfrac{2}{x^3} - \tfrac{30}{x^4} &= 0, && \text{multiplying by } x^4 \text{ gives} \\ 2x - 30 &= 0 && \text{simplifying gives} \\ x &= 15 \end{aligned}$$

(b) *Number Line*: We plug in 10 and 20 (anything between 0 and 15 and anything after 15) to the second derivative and find that $f''(10) = -0.001$ is negative and $f''(20) = 0.000625$ is positive. We summarize:



Summary: $x = 10$ gives a local maximum and $x = 15$ gives a point of inflection. And we should be able to roughly sketch the shape of the function (for a more accurate sketch, plug these values back into the original function to get the corresponding y values):