1. (13 pts) Box your final answer to each of the following.

(a) Let $g(t) = \sqrt{3 + \ln(5t - t^4)}$, find $g'(t)$.

$$\boxed{g'(t) = \frac{1}{2}\left(3 + \ln(5t - t^4)\right)^{-\frac{1}{2}} \cdot \frac{(5 - 4t^3)}{(5t - t^4)}}$$

(b) Find $\int \frac{5t}{3} - \frac{7}{8t} + \frac{6}{e^{5t}}\, dt.$ $= \int \frac{5}{3}t - \frac{7}{8}\frac{1}{t} + 6e^{-5t}\, dt$

$= \boxed{\frac{5}{6}t^2 - \frac{7}{8}\ln(t) - \frac{6}{5}e^{-5t} + C}$

(c) Evaluate $\int_1^{25} \frac{4}{\sqrt{x}}\, dx.$ $= 4 \cdot 2x^{1/2} \Big|_1^{25}$

$= 8\left(\sqrt{25} - \sqrt{1}\right)$

$= 8 \cdot (5 - 1) = \boxed{32}$

(d) Let $z = 3x^5 e^{2x} + y\ln(x) + \frac{4}{y^3}$, find BOTH the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= 15x^4 e^{2x} + 6x^5 e^{2x} + \frac{y}{x} \\ \frac{\partial z}{\partial y} &= \ln(x) - 12y^{-4} \end{aligned}}$$

2. (11 pts) Let $f(x) = 5x - 3x^2 + 1$.

(a) Write out, expand and *completely simplify* the formula, in terms of h, for

$$\frac{f(x+h) - f(x)}{h}.$$

$$\frac{[5(x+h) - 3(x+h)^2 + 1] - [5x - 3x^2 + 1]}{h}$$

$$= \frac{5x + 5h - 3(x^2 + 2xh + h^2) + \cancel{1} - \cancel{5x} + 3x^2 - \cancel{1}}{h}$$

$$= \frac{5h - 3x^2 - 6xh - 3h^2 + \cancel{3x^2}}{h}$$

$$= 5 - 6x - 3h$$

ANSWER: $\frac{f(x+h) - f(x)}{h} = \boxed{5 - 6x - 3h}$

(b) Find the slope of the secant line to $f(x)$ from $x = 3$ to $x = 3.5$.

$x = 3, \; h = 0.5$

$$\frac{f(3.5) - f(3)}{0.5} = 5 - 6(3) - 3(\tfrac{1}{2}) = 5 - 18 - 1.5$$
$$= 5 - 19.5$$

ANSWER: $\boxed{-14.5}$

(c) Find the slope of the tangent line to $f(x)$ at $x = 3$.

$f'(x) = 5 - 6x$

$f'(3) = 5 - 6(3) = 5 - 18 = -13$

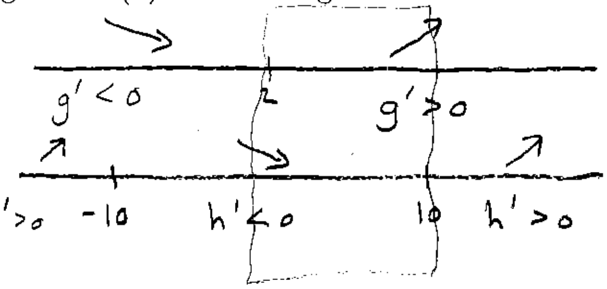
ANSWER: $\boxed{-13}$

3. (14 pts)

(a) Let $g(x) = 2x^2 - 8x + 12$ and $h(x) = \frac{4}{3}x^3 - 400x + 2$.
Find the longest interval on which $g(x)$ is increasing AND $h(x)$ is decreasing.

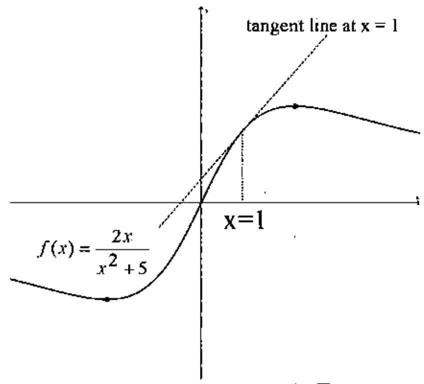
$g'(x) = 4x - 8 \stackrel{?}{=} 0 \iff x = 2$

$h'(x) = 4x^2 - 400 \stackrel{?}{=} 0 \Rightarrow x = \pm 10$



ANSWER: $x =$ 2 to $x =$ 10

(b) Consider $f(x) = \dfrac{2x}{x^2 + 5}$ (shown below).



i. Find $f'(x)$. (Hint: Quotient rule, check your work!).

$$f'(x) = \boxed{\frac{(x^2+5)(2) - 2x \cdot (2x)}{(x^2+5)^2}}$$

$$= \frac{2x^2 + 10 - 4x^2}{(x^2+5)^2} = \frac{10 - 2x^2}{(x^2+5)^2}$$

ii. Find the following:

A. The height of the graph at $x = 1$ is equal to $f(1) = \frac{2}{1+5} = \frac{2}{6} = \boxed{\frac{1}{3}}$

B. The slope of the graph at $x = 1$ is equal to $f'(1) = \frac{10-2}{(6)^2} = \frac{8}{36} = \boxed{\frac{2}{9}}$.

C. The equation for the tangent line at $x = 1$ is $\boxed{y = \frac{2}{9}(x-1) + \frac{1}{3}}$.
(This tangent line is show in the picture).

iii. You can see in the graph that there are two points (marked with black dots) where $f(x)$ has a horizontal tangent. Find the x-coordinates of both these points. (You can leave in exact form or give decimal approximations).

WANT $\frac{10 - 2x^2}{(x^2+5)^2} \stackrel{?}{=} 0 \Rightarrow 10 = 2x^2$

$5 = x^2$

$x = \pm\sqrt{5}$

(List both) $\boxed{x = \pm\sqrt{5}}$

4. (11 pts) The two parts below are not related.

(a) The function $h(x) = 8\ln(x) - 2x + 5$ has one critical number. Find the critical number of $h(x)$ and indiciate if it gives a local maximum, local minimum, or horizontal point of inflection. Show all your work and reasoning (some justification is required).

$$h'(x) = \frac{8}{x} - 2 \overset{?}{=} 0 \Rightarrow 8 - 2x \overset{?}{=} 0 \Rightarrow 8 = 2x$$
$$\Rightarrow x = 4$$

$$h''(x) = -\frac{8}{x^2} \quad \text{CONCAVE DOWN}$$

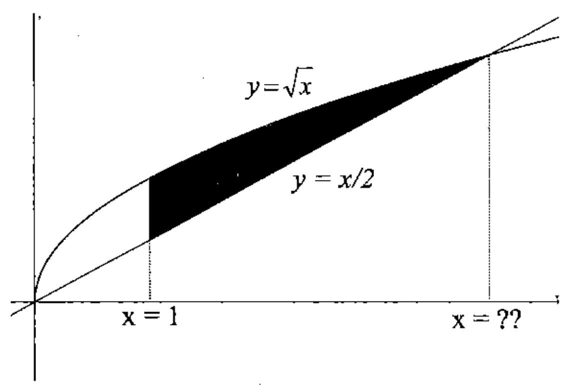
$\nearrow \quad \searrow$

$h' > 0 \quad 4 \quad h' < 0$

$x =$ 4

CIRCLE ONE: LOCAL MIN or LOCAL MAX or HORIZ. PT. OF INF.

(b) Find the area of the region bounded by $y = \sqrt{x}$ and $y = \frac{1}{2}x$ (shown below). Note: You will need to first find their intersection. (You may give your answer as a decimal to three digits after the decimal).

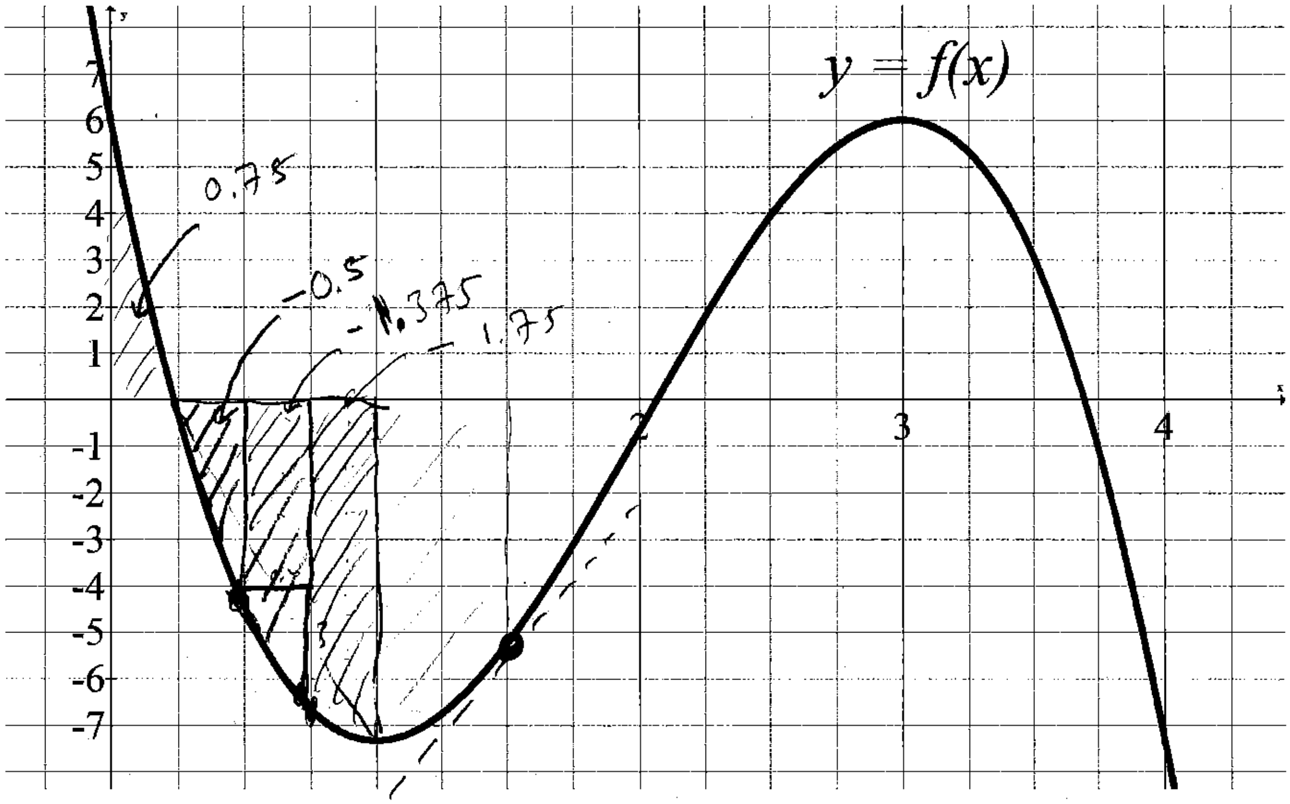


$$\left.\begin{aligned} \sqrt{x} &= \tfrac{1}{2}x \\ x &= \tfrac{1}{4}x^2 \end{aligned}\right\} \quad x = 4$$

$$\int_1^4 \sqrt{x} - \frac{1}{2}x \, dx$$
$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 \Big|_1^4$$
$$= \left[\frac{2}{3}(4)^{3/2} - \frac{1}{4}(4)^2\right] - \left[\frac{2}{3}(1)^{3/2} - \frac{1}{4}(1)^2\right]$$
$$= \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{14}{3} - 4 + \frac{1}{4} = \frac{56}{12} - \frac{48}{12} + \frac{3}{12}$$
$$= \frac{11}{12} \approx 0.917$$

Area $= \boxed{\frac{11}{12} \approx 0.917}$

5. (12 pts) Consider the graph of $y = f(x)$ below.



As precisely as possible, estimate your answer to the following questions using the graph.

For all parts, assume $A(m) = \int_0^m f(x)dx$.

(a) For each part below, circle the correct answer.

i. The value of $A(1.5)$ is:	POSITIVE	NEGATIVE	ZERO.
ii. The value of $f(1.5)$ is:	POSITIVE	NEGATIVE	ZERO.
iii. The value of $f'(1.5)$ is:	POSITIVE	NEGATIVE	ZERO.
iv. The value of $f''(1.5)$ is:	POSITIVE	NEGATIVE	ZERO.

(b) Find the value(s) of x at which $f'(x) = 0$ and $f''(x)$ is negative.

ANSWER: $x = \boxed{3}$ approx.

(c) Find the value(s) of x at which $A(x)$ has a local maximum.

$A'(x) = f(x) \overset{?}{=} 0$

ANSWER: $\boxed{x \approx 0.25, 3.7}$

(d) As accurately as possible, estimate the values of the following from the graph:

i. $A(1) = 0.75 - 0.5 - 1.375 - 1.75$

Approx. $\approx \boxed{-2.875}$

6, 0.75, 0.25

14 Boxes
$-$ 3.5 Boxes
10.5 Boxes

$10.5 \cdot 0.25$
≈ -2.625

ii. $A'(1) =$

$\boxed{= f(1) \approx -7.3}$

6. (14 pts) For your business you are given the selling price per item and the average cost per item as follows

SELLING PRICE :	$p = 66 - x$	dollar/item
AVERAGE COST :	$AC(q) = \frac{20}{x} + 81 - 9x + \frac{1}{3}x^2$	hundred dollars,

where x is in ***hundreds*** of items. Keep enough digits to be accurate to the nearest item.

(a) Find the functions for total revenue, total cost, marginal revenue and marginal cost.

$TR(x) =$ $66x - x^2$ $\qquad MR(x) =$ $66 - 2x$

$TC(x) =$ $20 + 81x - 9x^2 + \frac{1}{3}x^3$ $\qquad MC(x) =$ $81 - 18x + x^2$

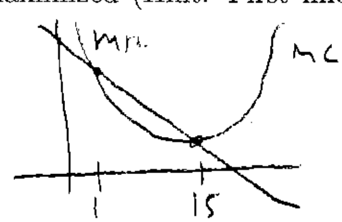
(b) Find the quantity q at which the second derivative $AC''(q)$ is equal to $\frac{5}{3}$. AND tell me if $AC(q)$ is concave up or concave down at this quantity.

$$AC'(q) = -\frac{20}{x^2} - 9 + \frac{2}{3}x$$
$$AC''(q) = \frac{40}{x^3} + \frac{2}{3} \stackrel{?}{=} \frac{5}{3}$$
$$\frac{40}{x^3} = 1$$
$$x^3 = 40$$
$$x = (40)^{1/3} \approx 3.4199$$

$\boxed{x = 3.42}$ hundred items

CIRCLE ONE: CONCAVE UP or CONCAVE DOWN or NEITHER

(c) Find the *selling price* that corresponds to when profit is maximized (Hint: First find the quantity that maximized profit).



$$81 - 18x + x^2 \stackrel{?}{=} 66 - 2x$$
$$x^2 - 16x + 15 \stackrel{?}{=} 0$$
$$(x-1)(x-15) \stackrel{?}{=} 0$$
$$x = 1, \quad x = 15$$

PROFIT MAXIMIZED HERE

$$p = 66 - 15 = 51$$

selling price = $\boxed{51}$ dollars/item

7. (14 pts) Let $z = f(x, y) = -x^2 + 6x - 3y^2 + 2y + 2xy + 12$.

(a) Write out the formulas for $f_x(x, y)$ and $f_y(x, y)$.

$f_x(x,y) =$ $-2x + 6 + 2y$ $\qquad$ $f_y(x,y) =$ $-6y + 2 + 2x$

(b) Find **all** critical points of $f(x, y)$.

$-2x + 6 + 2y \overset{?}{=} 0 \Rightarrow 6 + 2y = 2x$

$-6y + 2 + 2x \overset{?}{=} 0$

COMBINE $\Rightarrow -6y + 2 + (6 + 2y) \overset{?}{=} 0$

$8 - 4y \overset{?}{=} 0$

$y = 2 \Rightarrow$ $6 + 2(2) = 2x$

$10 = 2x$

$x = 5$

ANSWERS: $(x, y) =$ $\boxed{(5, 2)}$

(c) Use a partial derivative to approximate the value of $\frac{f(7.0001,2)-f(7,2)}{0.0001}$. (*i.e.* plug an appropriate point in the appropriate partial derivative like you did on the same problem in homework).

$f_x(7, 2) = -2(7) + 6 + 2(2) = -14 + 6 + 4$

ANSWER: $\boxed{-4}$

(d) Find the global minimum and maximum values of the one variable function $z = f(2, y)$ on the interval $y = 0$ to $y = 3$.

$z = f(2, y) = +4 + 12 - 3y^2 + 2y + 4y + 12$

$z = -3y^2 + 6y + 20 \longrightarrow z' = -6y + 6 \overset{?}{=} 0$

$y = 1$

at $y = 0$: $z = 20$

at $y = 1$: $z = -3 + 6 + 20 = 23$

at $y = 3$: $z = -3 \cdot 9 + 18 + 20 = -27 + 38 = 11$

ANSWER: Global Min Value: $z =$ 11

Global ~~Min~~ Max Value: $z =$ 23

8. (11 pts) A company manufactures two products, A and B. If x is the number of thousands of units of A and y is the number of thousands of units of B, then the total cost and total revenue in thousands of dollars are:

$$C(x, y) = 10x + 5y + x^2 + y^2 + xy$$
$$R(x, y) = 80x + 70y$$

The profit function has one critical point and the maximum profit occurs at this point. Find the maximum profit.

PROFIT $= (80x + 70y) - (10x + 5y + x^2 + y^2 + xy)$

$P(x,y) = 70x + 65y - x^2 - y^2 - xy$

$P_x = 70 - 2x - y \stackrel{?}{=} 0 \Rightarrow y = 70 - 2x$

$P_y = 65 - 2y - x \stackrel{?}{=} 0$

COMBINE $\Rightarrow$ $65 - 2(70 - 2x) - x \stackrel{?}{=} 0$

$65 - 140 + 4x - x \stackrel{?}{=} 0$

$3x = 75$

$x = 25$

$y = 70 - 50 = 20$

$P(25, 20) = 70(25) + 65(20) - (25)^2 - (20)^2 - (25)(20)$

$= 1750 + 1300 - 625 - 400 - 500$

$= 3050 - 1525$

$= 1525$

Maximum profit = 1525 thousand dollars which occurs when

$x =$ 25 thousand units of A and $y =$ 20 thousand units of B