1. (12 pts)

(a) Find the derivative of $f(x) = x^3 \ln(5x+1) - \frac{16}{3x^2} + 5$ $= x^3 \ln(5x+1) - \frac{16}{3}x^{-2} + 5$

$$\boxed{f'(x) = x^3 \frac{5}{5x+1} + 3x^2 \ln(5x+1) + \frac{32}{3}x^{-3}}$$

(b) Assume the selling price per item is given by $p = \frac{32}{3q+2}$. Find the marginal revenue at $q = 1$ items. (Hint: Write down TR first!)

$$TR(q) = \frac{32q}{3q+2}$$

$$MR(q) = \frac{(3q+2)(32) - (32q)(3)}{(3q+2)^2}$$

$$MR(1) = \frac{(5)(32) - (32)(3)}{5^2} = \frac{64}{25} = \boxed{2.56}$$

(c) Find the *equation for the tangent line* to $y = \sqrt{e^{5x} + 3}$ at $x = 0$.

$$y = (e^{5x} + 3)^{1/2} \Rightarrow y(0) = (1+3)^{1/2} = 2$$

$$y' = \frac{1}{2}(e^{5x}+3)^{-1/2} \cdot e^{5x} \cdot 5 \Rightarrow y'(0) = \frac{1}{2}(4)^{-1/2} \cdot 1 \cdot 5 = 5/4$$

$$\boxed{y = \frac{5}{4}x + 2}$$

2. (13 pts)

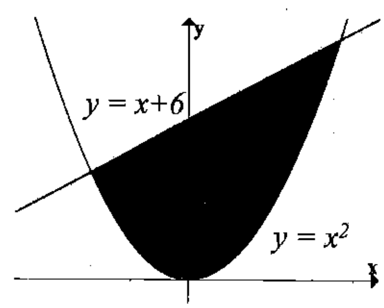
(a) Expand and integrate $\int x(3x+2)^2 dx$.

$$= \int x(9x^2+12x+4)\,dx$$
$$= \int 9x^3+12x^2+4x\,dx$$
$$= \boxed{\frac{9}{4}x^4+4x^3+2x^2+C}$$

(b) Compute $\int_1^9 \frac{5}{2\sqrt{x}}dx$ $= \int_1^9 \frac{5}{2}x^{-1/2}\,dx$

$$= 5x^{1/2}\Big|_1^9$$
$$= 5(3-1) = \boxed{10}$$

(c) Find the area of the region bounded by $y = x+6$ and $y = x^2$.
(Give your final answer as a decimal)



$$x^2 = x+6 \Rightarrow x^2-x-6=0$$
$$(x+2)(x-3)=0$$
$$x=-2,\ x=3$$

$$\int_{-2}^{3} x+6-x^2\,dx$$
$$= \frac{1}{2}x^2+6x-\frac{1}{3}x^3\Big|_{-2}^{3}$$
$$= \left(\frac{1}{2}(3)^2+6(3)-9\right)-\left(\frac{1}{2}(-2)^2+6(-2)-\frac{1}{3}(-2)^3\right)$$
$$= \frac{9}{2}+9-\left(2-12+8/3\right)$$
$$= 19+\frac{9}{2}-8/3 = \frac{125}{6} \approx \boxed{20.8\bar{3}}$$

3. (13 pts) You watch a balloon rise and fall. The height of the balloon (in feet) after t minutes is given by $A(t) = 18t - 3t^2 + 25$.

(a) Write out and *completely simplify* the formula, in terms of t and h, for the change in height from t to $t+h$:

$$A(t+h) - A(t)$$

$$\left[18(t+h) - 3(t+h)^2 + 25\right] - \left[18t - 3t^2 + 25\right]$$

$$= \cancel{18t} + 18h - 3(t^2 + 2th + h^2) + 25 - 18t + 3t^2 - 25$$

$$= 18h - \cancel{3t^2} - 6th - 3h^2 + 3t^2$$

ANSWER: $A(t+h) - A(t) =$ $18h - 6th - 3h^2$

(b) Find the average rate of ascent for the balloon from $t = 0.5$ min to $t = 2.5$ min. (include units)

$t = 0.5,\ h = 2 \Rightarrow A(2.5) - A(0.5) = 18(2) - 6(\tfrac{1}{2})(2) - 3(2)^2$

$= 36 - 6 - 12 = 18$

$$\frac{A(2.5) - A(0.5)}{2} = 9$$

ANSWER: 9 Units = $\frac{ft}{min}$

(c) Find the instantaneous rate of ascent for the balloon at $t = 5$ minutes. (include units)

$A'(t) = 18 - 6t$

$A'(5) = 18 - 6(5) = 18 - 30$

CHECK

$$\frac{A(t+h) - A(t)}{h} = 18 - 6t - 3h$$

$\nwarrow h \to 0$

ANSWER: -12 Units = ft/min

(d) Find the height of the balloon at the moment it changes from rising to falling. (inlude units)

$18 - 6t \overset{?}{=} 0 \Rightarrow t = 3$

$A(3) = 18(3) - 3(3)^2 + 25 = 52$

ANSWER: 52 Units = ft

4. (11 pts) The demand function for a product is $p = \dfrac{77}{x+1}$ and the supply function is $p = 2 + 0.5x$, where p is the price per item, in dollars/item, and x in the number of items.

(a) Find the price and quantity that correspond to market equilibrium.

$$\frac{77}{x+1} = 2 + 0.5x \Rightarrow 77 = (2+0.5x)(x+1)$$

$$\Rightarrow 77 = 2x + 0.5x^2 + 0.5x + 2$$

$$\Rightarrow 0 = 0.5x^2 + 2.5x - 75$$

$$\Rightarrow 0 = x^2 + 5x - 150$$

$$\Rightarrow 0 = (x-10)(x+15)$$

$$\boxed{x = 10} \qquad x = -15$$

$$p = \frac{77}{10+1} = 7$$

$x =$ 10 items

$p =$ 7 dollars/item

(b) Compute the supplier's surplus.

$$p_1 x_1 - \int_0^{x_1} 2 + 0.5x \, dx$$

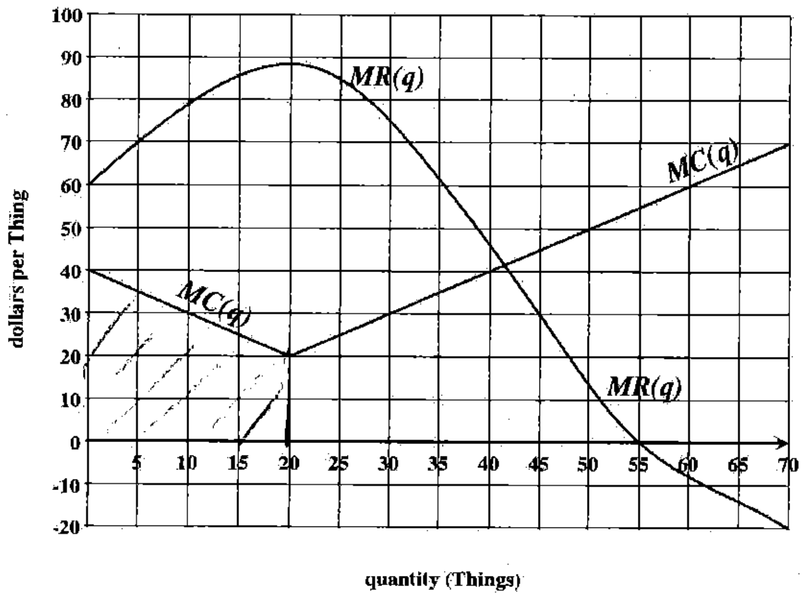
$$70 - \int_0^{10} 2 + 0.5x \, dx$$

$$= 70 - \left[2x + 0.25x^2 \Big|_0^{10}\right)$$

$$= 70 - [20 + 25] = 70 - 45 = 25$$

Supplier's surplus = 25 dollars

5. (16 pts) Below are the graphs of marginal revenue and marginal cost for selling Things:



Also assume that fixed cost is $FC = TC(0) = 25$ dollars.

(a) Estimate as accurately as possible from the graph:

i. $TC(20) =$ $\int_0^{20} mc(q)\,dq + 25 = \frac{1}{2}20(40+20)+25 = 600+25$ $= \boxed{625}$

ii. $TC'(30) =$ $MC(30) = \boxed{30}$

iii. $TC''(10) =$ $MC'(10) = \text{slope} = \boxed{-1}$ $\quad \frac{40-20}{0-20} = -1$

(b) Give the quantity at which each of the following occur (estimate from the graph):

i. The graph of MR has a local maximum at $q =$ $\boxed{20}$

ii. The graph of Profit has a local maximum at $q =$ $\boxed{\approx 41}$

iii. The graph of TR has a local maximum at $q =$ $\boxed{55}$

iv. On the interval $q = 5$ to $q = 33$, the absolute maximum of TC occurs at $q =$ $\boxed{33}$

(c) Give the longest interval of time over which the graph of TR is concave up.

MR increasing

$q =$ 0 to $q =$ 20

6. (12 pts) Consider the function

$$f(x) = \frac{x^3}{3} - 4x^2 + 12x.$$

(a) Find all critical numbers of $f(x)$ (there are two!).

$$f'(x) = x^2 - 8x + 12 \stackrel{?}{=} 0$$

$$(x-2)(x-6) \stackrel{?}{=} 0$$

$$x = 2 \text{ or } x = 6$$

$x =$ 2, 6

(b) Classify the critical numbers from the previous part as local max, or local min, or horizontal points of inflection. Clearly indicate your answers and show your reasoning.

f: $\nearrow$ $\searrow$ $\nearrow$

f': $+$ 2 $-$ 6 $+$

or

$$f''(x) = 2x - 8$$

$$f''(2) = -4 < 0 \quad \cap$$

$$f''(6) = 4 > 0 \quad \cup$$

$x = 2$ local max

$x = 6$ local min

(c) Consider the function $D(x) = \frac{f(x)}{x^2}$. Find the absolute minimum and absolute maximum values of $D(x)$ on the interval $x = 1$ to $x = 12$.

$$D(x) = \frac{1}{3}x - 4 + 12x^{-1}$$

$$D'(x) = \frac{1}{3} - 12x^{-2} = \frac{1}{3} - \frac{12}{x^2} \stackrel{?}{=} 0$$

$$\Rightarrow x^2 - 36 = 0 \Rightarrow x = \pm 6$$

$$D(1) = \frac{1}{3} - 4 + 12 = 8.\overline{3}$$

$$D(6) = 2 - 4 + 2 = 0$$

$$D(12) = 4 - 4 + 1 = 1$$

Absolute Minimum value = 0 which occurs at $x =$ 6

Absolute Maximum value = $8.\overline{3}$ which occurs at $x =$ 1

7. (12 pts) Let $z = f(x, y) = 14x - 12\ln(y) + \frac{2y^4}{x^3}$. $= 14x - 12\ln(y) + 2y^4x^{-3}$

(a) Write out the formulas for $f_x(x, y)$ and $f_y(x, y)$.

$f_x(x, y) =$ $14 - 6y^4x^{-4}$ $\qquad$ $f_y(x, y) =$ $-\frac{12}{y} + 8y^3x^{-3}$

(b) Use a partial derivative to approximate the value of $\frac{f(2.0001, 1) - f(2, 1)}{0.0001}$

x changing!

$$f_x(2,1) = 14 - \frac{6(1)^4}{(2)^4} = \frac{109}{8} = 13.625$$

ANSWER: 13.625

(c) If $y = 1$ is fixed, the function $g(x) = f(x, 1)$ is a one variable function of x. By showing appropriate calculations, answer the following questions:

i. Is $g(x)$ increasing, decreasing, or neither at $x = 3$?

$$g(x) = 14x + 2x^{-3}$$

$$g'(x) = 14 - 6x^{-4} = f_x(x,1)$$

$$g'(3) = 14 - \frac{6}{3^4} = 13.9\overline{25} > 0$$

ANSWER: (circle one) INCREASING DECREASING NEITHER

ii. Is $g(x)$ concave up, concave down, or neither at $x = 3$?

$$g''(x) = 24x^{-5} = \frac{24}{x^5}$$

$$g''(3) = \frac{24}{(3)^5} > 0$$

ANSWER: (circle one) CONCAVE UP CONCAVE DOWN NEITHER

8. (11 pts) A company manufactures two products, A and B. If x is in thousands of units of A and y is in thousands of units of B, then the total cost and total revenue in thousands of dollars are:

$$C(x, y) = 2x^2 - 2xy + y^2 - 8x - 10y + 11$$
$$R(x, y) = 8x + 6y$$

The profit function has one critical point and the maximum profit occurs at this point. Find the maximum profit.

$$P(x,y) = [8x+6y] - [2x^2-2xy+y^2-8x-10y+11]$$
$$= -2x^2+2xy-y^2+16x+16y-11$$

$$P_x = -4x+2y+16 \overset{?}{=} 0 \Rightarrow 2y = 4x-16$$
$$y = 2x-8$$

$$P_y = 2x-2y+16 \overset{?}{=} 0 \Rightarrow x-y+8=0$$

COMBINE!

$$x-(2x-8)+8=0$$
$$x-2x+8+8=0$$
$$-x+16=0$$
$$\boxed{x=16}$$

$$y = 2(16)-8$$
$$= 32-8$$
$$\boxed{y=24}$$

$$P(16,24)$$
$$= -2(16)^2+2(16)(24)-(24)^2 + 16(16)+16(24)-11$$
$$= 309$$

Maximum profit = 309 thousand dollars which occurs when

$x =$ 16 thousand units of A and $y =$ 24 thousand units of B