1. (10 pts)

(a) Find $f'(x)$ if $f(x) = \ln(x^3+1)e^{5x}$.

PRODUCT RULE!

$$\boxed{\begin{aligned} f'(x) &= \ln(x^3+1)\,e^{5x}\cdot 5 + e^{5x}\,\frac{1}{x^3+1}\,3x^2 \\ &= 5\ln(x^3+1)\,e^{5x} + \frac{3x^2e^{5x}}{x^3+1} \end{aligned}}$$

(b) Find $g_y(x,y)$ if $g(x,y) = \left(x^2 + \frac{2y^4}{3}\right)^{10} + \ln(x) + 7$. $= (x^2 + \frac{2}{3}y^4)^{10} + \ln(x) + 7$

$$\boxed{\begin{aligned} g_y(x,y) &= 10\left(x^2+\tfrac{2}{3}y^4\right)^9\cdot\tfrac{2}{3}\cdot 4y^3 \\ &= \frac{80}{3}\left(x^2+\tfrac{2}{3}y^4\right)^9 y^3 \end{aligned}}$$

(c) Find the equation of the tangent line to $h(x) = \frac{\sqrt{x+1}}{2} - \frac{27}{x^2}$ at $x = 3$.

$h(x) = \frac{1}{2}(x+1)^{1/2} - 27x^{-2}$

$h'(x) = \frac{1}{4}(x+1)^{-1/2} + 54x^{-3} = \frac{1}{4\sqrt{x+1}} + \frac{54}{x^3}$

$h(3) = \frac{\sqrt{3+1}}{2} - \frac{27}{3^2} = \frac{2}{2} - \frac{27}{9} = 1 - 3 = -2$

$h'(3) = \frac{1}{4\sqrt{3+1}} + \frac{54}{3^3} = \frac{1}{8} + 2 = 2.125$

$$\boxed{\begin{aligned} y &= 2.125(x-3) - 2 \\ y &= \tfrac{17}{8}(x-3) - 2 \end{aligned}}$$

2. (10 pts)

(a) Evaluate $\int \frac{3}{2x} + \frac{5}{3x^2} - \sqrt[3]{x}\, dx$

$= \int \frac{3}{2} \frac{1}{x} + \frac{5}{3} x^{-2} - x^{1/3} dx$

$$\boxed{\begin{aligned} &= \frac{3}{2}\ln(x) + \frac{5}{3}\frac{1}{-1}x^{-1} - \frac{3}{4}x^{4/3} + C \\ &= \frac{3}{2}\ln(x) - \frac{5}{3x} - \frac{3}{4}x^{4/3} + C \end{aligned}}$$

(b) A new financial company opens up and expects its **monthly** rate of income flow to be given by $r(t) = 3000e^{0.04t}$ dollars per month, where t is months after it opened. Find the total income in the first 3 years of operation.

3 years = 36 months

$\int_0^{36} 3000 e^{0,04t} dt$

$= \frac{3000}{0,04} e^{0,04t} \Big|_0^{36}$

$= 75000 e^{0,04(36)} - 75000 e^{0,04(0)}$

$= 75000 e^{1.44} - 75000$

$= 316552.19 - 75000$

$= \boxed{\$ 241,552.19}$

3. (11 pts) The demand function for a product is $p = 50 - x^2$ and the supply function is $p = 2 + 2x$, where p is the price per unit, in dollars, and x in the number of units.

(a) Find the price and quantity that correspond to market equilibrium.

$$50 - x^2 = 2 + 2x \longrightarrow x = \frac{-2 \pm \sqrt{4 - 4(-48)}}{2}$$

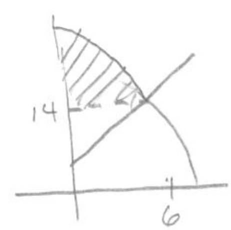
$$\Rightarrow \quad 0 = x^2 + 2x - 48 \quad \text{or} \quad = \frac{-2 \pm \sqrt{196}}{2} = \frac{-2 \pm 14}{2}$$

$$0 = (x+8)(x-6)$$

$$\boxed{x = 6} \quad \text{or} \quad \cancel{x = -8}$$

$$\left.\begin{array}{l} p = 50 - 6^2 \\ p = 2 + 2(6) \end{array}\right\} \boxed{p = \$14}$$

(b) Find the consumer's surplus.



$$\int_0^6 50 - x^2 \, dx - 6 \cdot 14$$

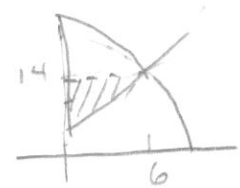
$$= \left(50x - \tfrac{1}{3}x^3 \Big|_0^6\right) - 84$$

$$= \left(50(6) - \tfrac{1}{3}(6)^3\right) - (0) - 84$$

$$= 300 - 72 - 84$$

$$= \boxed{\$144}$$

(c) Find the supplier's surplus.



$$6 \cdot 14 - \int_0^6 2 + 2x \, dx$$

$$84 - \left(2x + x^2 \Big|_0^6\right)$$

$$84 - \left((2(6) + (6)^2) - (0)\right)$$

$$84 - (12 + 36)$$

$$84 - 48 = \boxed{\$36}$$

4. (10 pts) Assume the height of a balloon is given by $A(t) = t^2 - 4t + 30$ where t is in minutes and height is in feet.

(a) Write out and completely simplify the expression $\dfrac{A(t+h) - A(t)}{h}$

$$\frac{[(t+h)^2 - 4(t+h) + 30] - [t^2 - 4t + 30]}{h}$$

$$= \frac{\cancel{t^2} + 2th + h^2 - 4t - 4h + 30 - \cancel{t^2} + 4t - 30}{h}$$

$$= \frac{2th + h^2 - 4h}{h}$$

$$= \boxed{2t + h - 4}$$

(b) Find the **average** rate of ascent over the interval from $t = 2$ to $t = 5$. (Give units)

$$\frac{A(5) - A(2)}{3} = ?$$

USE ORIGINAL OR FASTER TO USE PART (a) WITH $t = 2$ AND $h = 3$

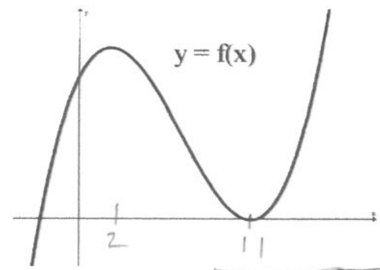
$$2(2) + 3 - 4 = \boxed{3 \ \text{ft}/\text{min}}$$

(c) Find the **instantaneous** rate of ascent at $t = 5$. (Give units)

$A'(t) = 2t - 4$ which matches part (a) as $h \to 0$ (a good check of your work!)

$$A'(5) = 2(5) - 4 = \boxed{6 \ \text{ft}/\text{min}}$$

5. (a) (7 pts) Let $f(x) = \frac{4}{3}x^3 - 26x^2 + 88x + 400$.
(A sketch of $f(x)$ is given).



i. Find the longest interval over which $f(x)$ is decreasing.

$f'(x) = 4x^2 - 52x + 88 \overset{?}{=} 0$

$\Rightarrow x^2 - 13x + 22 \overset{?}{=} 0$ or $x = \frac{52 \pm \sqrt{52^2 - 4(4)(88)}}{2(4)}$

$(x-11)(x-2) = 0$

$x = 11$ or $x = 2$

ANSWER: $x =$ 2 to $x =$ 11

ii. Find the x value(s) at which $f(x)$ has a point of inflection.

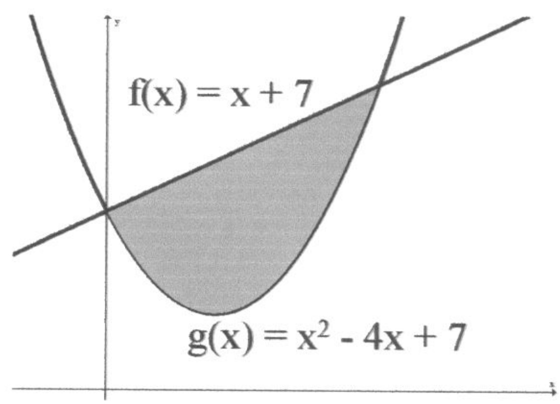
$f''(x) = 8x - 52 \overset{?}{=} 0$

$8x = 52$

$x = \frac{52}{8} = \frac{13}{2} = 6.5$

ANSWER: $x =$ 6.5

(b) (5 pts) Find the area of the shaded region between the given curves.
(Hint: First find where the functions intersect.)



$x^2 - 4x + 7 = x + 7$

$x^2 - 5x = 0$

$x(x-5) = 0$

$x = 0$ and $x = 5$

$\int_0^5 (x+7) - (x^2 - 4x + 7)\,dx$

$= \int_0^5 -x^2 + 5x\,dx$

$= -\frac{1}{3}x^3 + \frac{5}{2}x^2 \Big|_0^5$

$= \left(-\frac{1}{3}(5)^3 + \frac{5}{2}(5)^2\right) - (0)$

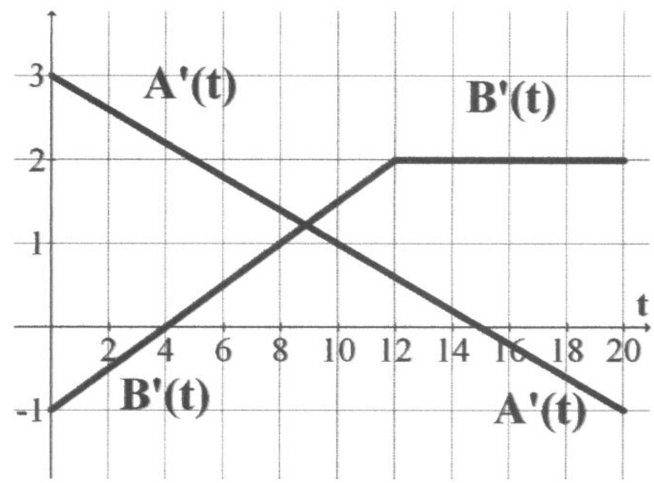
$= -\frac{125}{3} + \frac{125}{2} = -\frac{250}{6} + \frac{375}{6} = \frac{125}{6} \approx 20.8\bar{3}$

ANSWER: Area = $\frac{125}{6} = 20.8\bar{3}$

6. (12 pts)

Let $A(t)$ and $B(t)$ be the functions for the altitude of two Balloons, A and B. The graphs of **RATE OF ASCENT** for each balloon are given, where the t-axis is in minutes and the y-axis is in feet per minute.

You are told that both balloons start at an initial height of 60 ft, so $A(0) = 60$ and $B(0) = 60$. Accurately approximate answers to the following questions from the graph.



(a) (3 pts) Find the altitude of balloon A at time $t = 5$ minutes.

$A(5) - A(0) = \int_0^5 A'(t)dt = \frac{1}{2}(3+2)(5) = \frac{25}{2} = 12.5$ feet up

Since $A(0) = 60$ we have $A(5) = 60 + 12.5 = 72.5$

ANSWER: $A(5) =$ $\boxed{72.5}$ feet

(b) (3 pts) Find the longest interval of time when both the altitude of Balloon A and the altitude of Balloon B are rising (increasing).

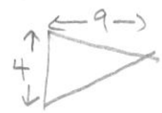
Need A' positive and B' positive

ANSWER: $t =$ 4 to $t =$ 15

(c) (3 pts) Let $D(t) = A(t) - B(t)$ be the vertical distance that Balloon A is above B. Find the time and value of $D(t)$ when it is largest.

At $t = 9$! (Before that A is getting farther above)

AREA BETWEEN $= \frac{1}{2}(4)(9) = 18$ feet



ANSWER: $t =$ 9 min | LARGEST DIST.= 18 feet

(d) (3 pts) Find the time, t, when balloon B will be 80 feet high.
(Hint: It might help to first know the height at $t = 12$)

At $t = 12$, height $= 60 + \int_0^{12} B'(t)dt$

$= 60 - \frac{1}{2}(1)(4) + \frac{1}{2}(2)(8)$

$= 60 - 2 + 8 = 66$ feet

So NOT TO HEIGHT 80 feet yet!

NEED TO GO 14 MORE FEET.

$2 \boxed{14} 2$

MUST BE 7 more units

$12 + 7 = 19$

ANSWER: $t =$ 19 min

7. (12 pts) You sell Things. The total cost is given by

$$TC(x) = 3x + \ln(x+1) + 10.$$

The marginal revenue is given by

$$MR(x) = 20 - 2x.$$

In the formulas, x is in thousands of Things, TC is in thousands of ~~Things~~, and MR is in dollars per Thing. Remember, as always, that $TR(0) = 0$.

In each problem below, your final answers should have enough digits to be accurate to the nearest Thing, or nearest dollar.

(a) (4 pts) Find the maximum value, in thousands of dollars, of total revenue.

$MR(x) = 20 - 2x = 0$ when $x = \frac{20}{2} = 10$ thousand things

$TR(x) = \int 20 - 2x\,dx$

$TR(x) = 20x - x^2 + C$

$TR(0) = 0 \Rightarrow 20(0) - (0)^2 + C = 0 \Rightarrow C = 0$

$TR(x) = 20x - x^2$

$TR(10) = 20(10) - (10)^2 = 200 - 100 = 100$

ANSWER: $\boxed{100}$ thousand dollars

(b) (4 pts) Find the quantity at which profit is maximum.

$MC(x) = 3 + \frac{1}{x+1} \stackrel{?}{=} 20 - 2x = MR(x)$

$\frac{1}{x+1} = 17 - 2x$

$1 = (17 - 2x)(x+1)$

$1 = -2x^2 + 15x + 17$

$0 = -2x^2 + 15x + 16$

$x = \frac{-15 \pm \sqrt{15^2 - 4(-2)(16)}}{2(-2)}$

$x = \frac{-15 \pm \sqrt{353}}{-4}$

$x = \frac{-15 \pm 18.78829423}{-4}$

$x = 8.44707$ or $x = -0.94707$

ANSWER: $q = \boxed{8.447}$ thousand Things

(c) (4 pts) Recall that average cost is given by $AC(q) = \frac{TC(q)}{q}$. By making appropriate calculations with $AC(q)$ and its derivatives, determine if $AC(q)$ is increasing, decreasing, or neither at $x = 1$.

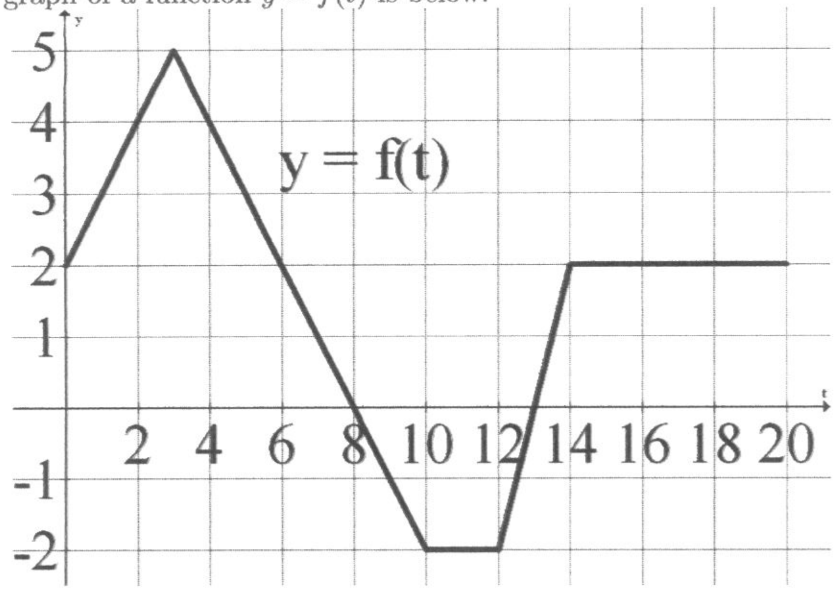
$AC(x) = \frac{3x + \ln(x+1) + 10}{x}$

$AC'(x) = \frac{x\left(3 + \frac{1}{x+1}\right) - (3x + \ln(x+1) + 10)(1)}{x^2}$

$AC'(1) = \frac{(1)\left(3 + \frac{1}{2}\right) - (3(1) + \ln(2) + 10)}{(1)^2} = -\ln(2) - 9.5 < 0$

ANSWER: (circle one) INCREASING DECREASING NEITHER

8. (12 pts) The graph of a function $y = f(t)$ is below.



Using the graph above, we define a new function

$$A(m) = \int_0^m f(t)\,dt$$

(a) Compute the following:

- $A'(6)$ $= f(6) = 2$

ANSWER: $A'(6) = \boxed{2}$

- $A''(6)$ $= f'(6) = \text{slope} = \frac{2-0}{6-8} = \frac{2}{-2} = -1$

TWO POINTS (6, 2), (8, 0)

ANSWER: $A''(6) = \boxed{-1}$

(b) Compute the value of $\int_8^{20} f(t)\,dt$

8 to 10 $\quad -\frac{1}{2}(2)(2) = -2$

10 to 12 $\quad -2(2) = -4$

12 to 14 IS ZERO!

14 to 20 $= 6 \cdot 2 = 12$

SIGNED AREA $= -2 - 4 + 0 + 12 = 6$

ANSWER: $\int_8^{20} f(t)\,dt = \boxed{6}$

(c) Find all values of m between 0 and 20 at which $A(m)$ has a local minimum.

$A'(m)$ CHANGE FROM NEGATIVE TO POSITIVE!

ANSWER: $m = \boxed{13}$

(d) Find the global maximum value of $A(m)$.

$A(8) = \frac{1}{2}(2+5)3 + \frac{1}{2}(5) \cdot 5$

$= \frac{21}{2} + \frac{25}{2}$

$= \frac{46}{2} = 23$

ANSWER: 'Max output from $A(m)$' $= \boxed{29}$

SINCE $\int_8^{20} f(t)\,dt$ IS POSITIVE, THE MAX WILL OCCUR AT $m = 20$. WITH A VALUE OF $23 + 6 = 29$

9. (11 pts) Your company produces and sells Seahawks gloves and Mariner hats. In a given month, let x be in hundreds of gloves produced and sold and let y be in hundreds of hats produced and sold. The profit for the month in hundreds of dollars is given by:

$$P(x,y) = 8x + 4xy - 5x^2 - y^2 - 3 \text{ hundred dollars.}$$

(a) Compute the partial derivatives of P.

$P_x(x,y) =$ $8 + 4y - 10x$

$P_y(x,y) =$ $4x - 2y$

(b) Suppose you currently produce and sell $x = 3$ hundred gloves and $y = 4$ hundred gloves. What would increase your profit more: Selling one more glove or selling one more hat? (Circle one and give a justification that involves partial derivatives)

$P_x(3,4) = 8 + 4(4) - 10(3) = 8 + 16 - 30 = -6 \leftarrow$ one more glove decreases profit!

$P_y(3,4) = 4(3) - 2(4) = 12 - 8 = 4 \leftarrow$ one more hat increases profit by \$4

(c) You are told that the maximum of profit occurs at the critical point. Find the critical point of profit and give the maximum profit value.

(i) $8 + 4y - 10x = 0$

(ii) $4x - 2y = 0 \Rightarrow 4x = 2y \Rightarrow y = 2x$

(i) AND (ii) $\Rightarrow$ $8 + 4(2x) - 10x = 0$

$8 + 8x - 10x = 0$

$8 - 2x = 0$

$8 = 2x$

$\boxed{x = 4}$

$\boxed{y = 2(4) = 8}$

$P(4,8) = 8(4) + 4(4)(8) - 5(4)^2 - (8)^2 - 3 = \boxed{13}$

$=$

Critical point: $(x, y) =$ $(4, 8)$

'Max output from profit' $=$ 13 hundred dollars