Chapter 14 Overview

In this chapter we discussed the basics of calculus with multivariable functions. A **multivariable function** has multiple inputs. Many real situations involve multivariable functions.

For example: Assume you sell x hats at a price of \$15 per hat and you sell y coats at a price of \$40 per coat. Then your total revenue is given by $TR(x,y) = 15x + 40y$. In this situation, your cost and profit functions would also be multivariable. If you are told that $TC(x,y) = 5x + 2y + \frac{1}{2}x^2 + y^2 + 13$, then your profit would be $P(x,y) = TR(x,y) - TC(x,y) = 10x + 38y - \frac{1}{2}x^2 - y^2 - 13$. That's a simple example of a multivariable function.

There are many, many other examples in business, economics, and industry where multiple inputs effect the output from a function. So one question is: How can we use our calculus tools to analyze such a function?

Partial Derivatives

IDEA: If we fix all the variables except one, then we can treat the problems as a one variable problem and do derivatives like we usually do. That will give the rate relative to that one variable.

1. More formally, if $z = f(x,y)$, then we define the following two partial derivatives

$$\frac{\partial z}{\partial x} = f_x(x,y) = \text{`what the secant slope } \frac{f(x+h,y)-f(x,y)}{h} \text{ approaches as } h \to 0\text{'}$$

$$\frac{\partial z}{\partial y} = f_y(x,y) = \text{`what the secant slope } \frac{f(x,y+h)-f(x,y)}{h} \text{ approaches as } h \to 0\text{'}$$

   Notice that in the definition of $f_x(x,y)$ it is the x value that is being changed (so we are measuring the effect of a small change in x), and in the definition $f_y(x,y)$ the y value is being changed (so we are measuing the effect of a small change in y. In other words (remember in class that I showed you a 3D visualization of multivariable function and I discussed the x and y direction slopes and how they can be different):

   $f_x(x,y) =$ 'rate of change of $f(x,y)$ with respect to x' = 'slope of the tangent line in the x-direction'

   $f_y(x,y) =$ 'rate of change of $f(x,y)$ with respect to y' = 'slope of the tangent line in the y-direction'

2. The marginal value perspective. As we did with MR and MC for one variable functions, we can roughly interpret this rate as the change that results from an increase of one unit. That is

$$f_x(x,y) \approx \frac{f(x+1,y)-f(x,y)}{1} \quad \text{and} \quad f_x(x,y) \approx \frac{f(x,y+1)-f(x,y)}{1}$$

   Using our example from above when $TR(x,y) = 15x + 40y$, we see that

$$TR_x(x,y) \approx TR(x+1,y) - TR(x,y) = [15(x+1) + 40y] - [15x + 40y] = 15$$

   'Revenue goes up \$15 dollars when you sell one more hat'

$$TC_y(x,y) \approx TR(x,y+1) - TR(x,y) = [15x + 40(y+1)] - [15x + 40y] = 40$$

   'Revenue goes up \$40 dollars when you sell one more coat'

   (Aside: Since the functions are linear here the derivative is actually exactly equal to this value as we will see in a moment when we start computing partial derivatives).

In general, the partial derivative will be close to any small 'secant' slope change for the function. Meaning, for example:

$$f_x(x,y) \approx \frac{f(x+0.001,y)-f(x,y)}{0.001} \quad \text{and} \quad f_y(x,y) \approx \frac{f(x,y+0.001)-f(x,y)}{0.001}$$

And the smaller the amount is that we are adding on, the closer we get to the actual derivative value.

3. To compute partial derivatives:

   (a) Identify which variable is fixed (for $f_x(x,y)$ we have y is fixed, and for $f_y(x,y)$ we have x is fixed). Go through your function and identify what is being treated as a constant. Distinguish in your mind between constants that are not attached to the variable (so their derivatives will be zero) and constants that are next to a variable (so they are coefficients and they 'come along for the ride').

   (b) Differentiate with respect to the desired variable using all your one variable calculus rules.

   As you start to do this, it might help you to actually plug in particular values for the fixed letter. Meaning, if you are having trouble finding $f_x(x,y)$, then on a scratch sheet plug in $y=1$ and do the derivative with the particular constant $y=1$ in place of y. However you are doing that derivative is the same way you will do it when the y is left there (meaning go back at the end and replace the 1's with y's again).

   Let's do some examples (find the partial derivatives):

   (a) $f(x,y) = x^3 + y^2 - 13x + 10$

   ANSWER:
   For $f_x(x,y)$, treat y as fixed (for example if $y=1$, we get $z = x^3 + 1^2 - 13x + 10$, think about what it's derivative would be). We get

   $$f_x(x,y) = 3x^2 - 13.$$

   For $f_y(x,y)$, treat x as fixed (for example if $x=1$, we get $z = 1^3 + y^2 - 13(1) + 10$, think about what it's derivative would be). We get

   $$f_y(x,y) = 2y.$$

   (b) $f(x,y) = 10x^4y^3 - x^5$

   ANSWER:
   For $f_x(x,y)$, treat y as fixed (for example if $y=1$, we get $z = 10x^4(1)^3 - x^5$). We get

   $$f_x(x,y) = 40x^3y^3 - 5x^4. \quad \text{(notice that } y^3 \text{ was a coefficient here)}$$

   For $f_y(x,y)$, treat x as fixed (for example if $x=1$, we get $z = 10(1)^4y^3 - (1)^5$). We get

   $$f_y(x,y) = 30x^4y^2. \quad \text{(notice that } x^4 \text{ was a coefficient here)}$$

(c) $f(x, y) = (x^4 + y^5)^{10}$

ANSWER:
For $f_x(x, y)$, treat y as fixed (for example if $y = 1$, we get $z = (x^4 + 1^5)^{10}$). We get

$$f_x(x, y) = 10(x^4 + y^5)^9 \cdot 4x^3. \quad \text{(Regular old chain rule)}$$

For $f_y(x, y)$, treat x as fixed (for example if $x = 1$, we get $z = (1^4 + y^5)^{10}$). We get

$$f_y(x, y) = 10(x^4 + y^5)^9 \cdot 5y^4. \quad \text{(Again, regular old chain rule)}$$

(d) $f(x, y) = e^{2x} + \ln(y) - 7$

ANSWER:
For $f_x(x, y)$, treat y as fixed (for example if $y = 1$, we get $z = e^{2x} + \ln(1) - 7$). We get

$$f_x(x, y) = e^{2x} \cdot 2. \quad \text{(Note that } \ln(y) \text{ is just a fixed number, which has derivative zero!)}$$

For $f_y(x, y)$, treat x as fixed (for example if $x = 1$, we get $z = e^2 + \ln(y) - 7$). We get

$$f_y(x, y) = \frac{1}{y}. \quad \text{(Again, note that } e^2 \text{ is just a fixed number, which has derviative zero!)}$$

(e) $f(x, y) = \ln(x^5 + y^3)$

ANSWER:
For $f_x(x, y)$, treat y as fixed (for example if $y = 1$, we get $z = \ln(x^5 + 1^3)$). We get

$$f_x(x, y) = \frac{1}{x^5 + y^3} \cdot 5x^4. \quad \text{(Regular old chain rule)}$$

For $f_y(x, y)$, treat x as fixed (for example if $x = 1$, we get $z = \ln(1^5 + y^3)$). We get

$$f_y(x, y) = \frac{1}{x^5 + y^3} \cdot 3y^2 \quad \text{(Again, regular old chain rule)}$$

4. Going back to our original example $TR(x, y) = 15x + 40y$, $TC(x, y) = 5x + 2y + \frac{1}{2}x^2 + y^2 + 13$, and $P(x, y) = 10x + 38y - \frac{1}{2}x^2 - y^2 - 13$, we get

$$TR_x(x, y) = 15 \quad TC_x(x, y) = 5 + x \quad P_x(x, y) = 10 - x$$

$$TR_y(x, y) = 40 \quad TC_y(x, y) = 2 + 2y \quad P_y(x, y) = 38 - 2y$$

Let's stop and interpret these briefly so we don't lose sight of what they represent.

(a) The TR_x and TR_y derivatives are constant. So no matter what the values of x and y are, the revenue goes up \$15 dollars if one more unit of x is sold and the revenue goes up \$40 if one more unit of y is sold.

(b) The TC_x and TC_y derivatives are not constant. So let's look at a particular example. How about $(x, y) = (2, 3)$ (you are producing 2 hats and 3 coats):
We get $TC_x(2, 3) = 5 + 2 =$ \$7. So if we produce 2 hats and 3 coats, then it will cost \$7 more to produce one more hat.
And $TC_y(2, 3) = 2 + 2(3) =$ \$8. So if we produce 2 hats and 3 coats, then it will cost \$8 more to produce one more coat.

(c) Also the P_x and P_y derivatives are not constant. So let's look at a different particular example. How about $(x, y) = (12, 5)$ (you are producing and selling 12 hats and 5 coats):
We get $P_x(12, 5) = 10 - 12 =$ -\$2. So if we produce 12 hats and 5 coats, then the profit will go down \$2 if you produce and sell one more hat.
And $P_y(12, 5) = 38 - 2(5) =$ \$28. So if we produce 12 hats and 5 coats, then the profit will go up \$28 if you produce and sell one more coat.

Critical Points
A point $(x, y) = (a, b)$ is a *critical point* of the function $f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

For a multivariable function, each critical point corresponds to an input that gives a local maximum, local minimum, or neither (this last one is called a saddle point in the regular multivarialbe calculus class, but you don't have to know that term). These are the points when there is a horizontal tangent in both the x and y directions (the rates are both zero). In Math 112, for time reasons, we do not discuss the methods for classifying our critical points. But you should know that local maximum and minimum values occur at these points. And if we tell you that the maximum is one of the critical points, then you should find the critical points, then plug them into the original $f(x, y)$ to see what gives the biggest output.

Finishing up our example from the beginning. Assume I tell you that the maximum profit occurs at a critical point. We know that $P_x(x, y) = 10 - x$ and $P_y(x, y) = 38 - 2y$. To find the critical points, we must solve the system of equations

$$\begin{aligned} 10 - x &= 0 \\ 38 - 2y &= 0 \end{aligned}$$

Solving each gives $x = 10$ and $y = 19$. So to maximize the profit we must produce and sell 10 hats and 19 coats, which gives a profit of $P(10, 19) = 10(10) + 38(19) - \frac{1}{2}(10)^2 - (19)^2 - 13 = \$398$.

Another finding critical points example: This is a problem from an old final.

Question: Find the critical points of $f(x, y) = -5x^2 - 5x - 3y^2 + 4y + 10xy + 18$.

Answer:

(a) Step 1: Find the partial derivatives:

$$f_x(x, y) = -10x - 5 + 10y \text{ and } f_y(x, y) = -6y + 4 + 10x.$$

(b) Step 2: Set them both equal to zero, combine and solve:

$$\begin{aligned} (i) \quad & -10x - 5 + 10y &= 0 \\ (ii) \quad & -6y + 4 + 10x &= 0 \end{aligned}$$

Note: There are two equally good approaches to combining a system of equations.

i. Option 1: Solve for one variable in one equation and substitute it into the other equation. Let me do that here.
From equation (i), we get $10y = 10x + 5$ which becomes $y = x + \frac{1}{2}$.
Substituting this fact into equation (ii) gives

$$-6\left(x + \frac{1}{2}\right) + 4 + 10x = 0 \Rightarrow -6x - 3 + 4 + 10x = 0 \Rightarrow 4x = -1$$

Thus, $x = -\frac{1}{4}$.
Then we can substitute back into any of the previous equations to get y, $y = x + \frac{1}{2} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$.
So the only critical point is $(x, y) = \left(-\frac{1}{4}, \frac{1}{4}\right)$

ii. Option 2: You can add or subtract the equations. This works well if you can get the coefficients to match up. In this case, we already had $-10x$ in the first equation and $10x$ in the second equation, so it makes sense to add the equations together. Here we do it:
Adding corresponding sides of equations (i) and (ii) give $-1 + 4y = 0$, so $y = \frac{1}{4}$.
Now you have to substitute this back into either (i) or (ii) (or both to check your work) to get x.
From (i), we have $-5 + 10y = 10x$, so $x = -\frac{1}{2} + y = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$.
So we get the same answer! You don't need to do both methods, just pick one that works well for you.