Solving Equations Brief Overview

One of the main algebraic difficulties students have at this point in the course is solving equations (in almost every question you are solving when the derivatives are equal to zero). There is no way I can cover all of algebra in this short sheet, but let me discuss some common situations and errors.
First, a bit of terminology is necessary:

- A mathematical *expression* is a formula involving numbers and variables (but there is NO equals sign). For example: $5x + 20$ and $e^x + 10\sqrt{x^2 - 1}$ are expressions. We don't solve expressions, that doesn't make sense.

- A *function* is an expression that we can evaluate at a particular variable. In which case, we typically give the function a name (we say we are defining the function). For example: $f(x) = 5x + 20$ or $g(x) = e^x + 10\sqrt{x^2 - 1}$ are function definitions. On the left is a function name and an indication of the input $f(x)$ or $g(x)$ and the equal sign here is being use to tell us the formula we use to compute that rule. We don't solve functions, that doesn't make sense. So even though there is an equal sign, we aren't ever solving $f(x) = 5x + 20$. That is just the definition of a rule for how to compute $f(x)$.

- An *equation* sets two expressions equal to each other. In this case we are trying to solve. For example $5x + 20 = 14$ or $x^2 = 2x + 5$.

Hopefully that is clear. So it only makes sense to be solving an equation if there is:

- An equal sign in each step! (Don't let the equal sign disappear!).

- Expressions (numbers and/or variables) on both sides of the equal sign at each step!

Solving 101

The goal in solving an equation is to get the variable by itself. We do this using a series of inverses. Here are all the basic inverses and how we use them:

The Inverses

Addition/Subtraction:
To solve $x + 5 = 7$, we subtract 5 from both sides to get $x = 2$.
And to solve $x - 3 = 10$, we add 3 to both sides to get $x = 13$.
Multiplication/Division:
To solve $6x = 30$, we divide 6 from both sides to get $x = 5$.
And to solve $\frac{x}{7} = 11$, we multiply 7 to both sides to get $x = 77$.
Powers/Roots:
To solve $x^3 = 10$, we take the cube root of both sides to get $x = 10^{1/3}$.
And to solve $x^{1/5} = 2$, we take the 5th power of both sides to get $x = 2^5$.
Important Notes about powers/roots:

- The bottom of a fractional exponent is a root and the top of a fractional exponent is a power. So for example, $x^{3/4}$ is asking you to take the 3rd power and 4th root of x. So if you are solving $x^{3/4} = 10$, then the inverse would be taking the 3rd root and 4th power, to get $x = 10^{4/3}$.
- When you start with an EVEN power, and you take the root of both sides, you must include $\pm$ to account for both possible solutions. For example, to solve $x^{10} = 3$, you should write $x = \pm 3^{1/10}$.

Exponentials/Logarithms:
To solve $e^x = 7$, we take the natural logarithm of both sides to get $x = \ln(7)$.
And to solve $\ln(x) = 20$, we take the exponential of both sides to get $x = e^{20}$.
Important Note about logs: Using the rules that $\ln(a^b) = b\ln(a)$ and $\ln(ab) = \ln(a) + \ln(b)$, you can solve many more equations that have other bases as well. For example, to solve $3^x = 40$, you take the natural logarithm of both sides to get $\ln(3^x) = \ln(40)$, then rewrite $x\ln(3) = \ln(40)$, then divide to get $x = \ln(40)/\ln(3)$.

General Strategies and Special Cases

1. If there are fractions, clear the denominator. Meaning multiply everything on both sides by the the denominator to get rid of it.
   For example, to solve $\frac{1}{x} - \frac{10}{x^2} = 0$. Start by multiplying by everything by x^2 which gives $x - 10 = 0$, so $x = 10$. Always do this first!!!
2. Get all your x's to the same side from the beginning. Then try to factor, combine or simplify your expressions involving x if you can.
   For example, to solve $x^2 - 3x = 0$, I would look to factor to get $x(x - 3) = 0$, so that my solution is $x = 0$ or $x = 3$.
3. If you end up with a quadratic equation that has all three terms and you can't factor, then you use the quadratic formula.
   For example, to solve $x^2 - 10x + 2 = 0$, I tried to factor but I couldn't quickly see a way to factor so I used $x = \frac{10 \pm \sqrt{100 - 4(2)}}{2}$.
4. Anything we ask you to solve can be solve by combinations of the simple rules above. Anything beyond these simple rules requires a numerical solution (done with a computer or calculator, which we don't ask you to do).